Published by Institute of Physics Publishing for SISSA/ISAS
Received: *November 18, 2003*
Revised: *January 6, 2004*
Accepted: *January 8, 2004*

The boundary Weyl anomaly in the $\mathcal{N} = 4$ SYM/type-IIB supergravity correspondence

Paul Mansfield
Department of Mathematical Sciences, University of Durham
South Road Durham, DH1 3LE, England, U.K.
E-mail: P.R.W.Mansfield@durham.ac.uk

David Nolland and Tatsuya Ueno
Department of Mathematical Sciences, University of Liverpool
Liverpool, L69 3BX, England, U.K.
E-mail: nolland@liv.ac.uk, ueno@liv.ac.uk

Abstract: We give a complete account of the Schrödinger representation approach to the calculation of the Weyl anomaly of $\mathcal{N} = 4$ SYM from the AdS/CFT correspondence. On the AdS side, the $1/N^2$ correction to the leading order result receives contributions from all the fields of type-IIB Supergravity, the contribution of each field being given by a universal formula. The correct matching with the CFT result is thus a highly non-trivial test of the correspondence.

Keywords: Anomalies in Field and String Theories, AdS-CFT and dS-CFT Correspondence, Supergravity Models.

 http://jhep.sissa.it/archive/papers/jhep012004013/jhep012004013.pdf

Contents

1. Introduction: the basic calculation

When Super-Yang-Mills theory is coupled to a non-dynamical, external metric, g_{ij}, the Weyl anomaly, $\mathcal{A}$, is the response of the 'free-energy' (i.e. the logarithm of the partition function), F, to a scale transformation of that metric. This is a quantum effect, since the classical theory is scale-invariant, but the one-loop result is exact because supersymmetry prevents higher loops from contributing. The result is proportional to $N^2 - 1$ and Henningson and Skenderis showed [1] that the N^2 part is correctly reproduced by a tree-level calculation in five-dimensional gravity confirming the Maldacena conjecture to leading order in large-N. Reproducing quantum effects in a gauge theory from classical gravity is itself truly remarkable but to go beyond the leading order and reproduce the -1 requires much more than just classical gravity, it needs the computation of Superstring loops. This is a stringent test of the details of the conjecture because although the graviton alone is responsible for the leading order result, all the species of fields in IIB Supergravity contribute at subleading order. The purpose of this paper is to show that the subleading contribution to the Weyl anomaly of Super-Yang-Mills theory is indeed obtained from quantum loops in Supergravity, confirming the Maldacena conjecture to this order.

We will compute the contribution to the Weyl anomaly of the boundary theory coming from one-loop IIB Supergravity compactified on S^5 propagating in a five-dimensional AdS space with a boundary that is an Einstein manifold, using an action that describes one-loop fluctuations. At this order the Kaluza-Klein modes are free and each independent contribution is the logarithm of the determinant of a five-dimensional operator. The standard approach to calculating these requires the use of heat-kernels on the five-dimensional manifold near the boundary. Instead we will adopt a hamiltonian approach that treats the boundary as a quantisation surface and so uses four-dimensional operators. We will show that the boundary Weyl anomaly of each bulk field has a universal form proportional to the product $(\Delta - 2)\, a_2$ where Δ is a scaling dimension in the boundary theory and a_2 is a Seeley-de Witt coefficient in the expansion of the heat-kernel of the four-dimensional operator. In contrast to the five-dimensional heat-kernels the four-dimensional ones were thoroughly studied a long time ago, and the appropriate asymptotic expansions tabulated. The Kaluza-Klein multiplets lead to four-dimensional operators describing particle of spins 0, 1/2, 1, 3/2, 2 but when the (regulated) sum over all the modes is done it reduces to an expression involving only those spins associated with the $\mathcal{N} = 4$ boundary Yang-Mills theory with exactly the coefficients that confirm the Maldacena conjecture to this sub-leading order in large-N.

The metric for $d+1$-dimensional Anti-de Sitter (AdS_{d+1}) space can be written

$$ds^2 = G_{\mu\nu}\, dX^\mu\, dX^\nu = dr^2 + z^{-2}\eta_{ij}dx^i dx^j\,, \qquad z = \exp\left(\frac{r}{l}\right), \tag{1.1}$$

with η_{ij} the d-dimensional Minkowski metric, and l a constant. The Riemann tensor is

$$R_{\mu\nu\lambda\rho} = -\frac{1}{l^2}\left(G_{\mu\lambda}G_{\nu\rho} - G_{\nu\lambda}G_{\mu\rho}\right), \tag{1.2}$$

which leads to the (d+1)-dimensional Einstein equation

$$R_{\mu\nu} = -\frac{d}{l^2}G_{\mu\nu}\,, \tag{1.3}$$

with the cosmological constant $\Lambda = -d(d-1)/2l^2$ and $R = -d(d+1)/l^2$. The boundary of this space occurs at $r = -\infty$ and at $r = \infty$ (which is just a point because the 'warp-factor', z^{-2} vanishes there). Following [1] we will treat this boundary as though it occurred at $z = \exp(r_0/l) \equiv \tau$ and send the cut-off, τ, to zero at the end of our calculations, so the metric restricted to the boundary is η_{ij}/τ^2.

Maldacena conjectured an equivalence between type-IIB String Theory compactified on $AdS_5 \times S^5$ (the bulk theory) and 4-dimensional Super-Yang-Mills theory in Minkowski space (the boundary of AdS_5) with gauge-group SU(N). The string compactification is driven by the presence of N D3-branes which generate a 5-form flux. Performing the functional integral for the Superstring theory with the fields taking prescribed values on the boundary of AdS_5 is meant to reproduce the generating functional of Green functions for operators, Ω, in the Yang-Mills theory. The identification between the boundary fields and the various Ω has been made on the basis of symmetry for many operators, and there is a precise relationship between the couplings in the two theories. After a Wick rotation the conjecture may be written as

$$\int \mathcal{D}\Phi\, e^{-S_{IIB}}\Big|_{\Phi(r=-\infty)=\hat{\Phi}} = \int \mathcal{D}A\, e^{-S_{YM}+\int d^4x\,\hat{\Phi}\Omega(A)} . \tag{1.4}$$

This is only a formal statement since the left-hand-side written in terms of the string fields of IIB Superstring theory is not well-defined. (Nonetheless, an observation that will be crucial later is that this ill-defined functional of the boundary fields, $\Psi[\hat{\Phi}]$ corresponds to Feynman's construction of the vacuum wave-functional.) Taking Ω as the stress-tensor of the gauge theory allows the source $\hat{\Phi}$ to be interpreted as a perturbation to the Minkowski metric, so that in the absence of other sources the right-hand-side becomes the partition function for Super-Yang-Mills theory in the perturbed metric, g_{ij}, whose logarithm we denote by $F[g]$. The Weyl anomaly, $\mathcal{A}$, is then $\delta F = \int d^4x\,\sqrt{g}\delta\sigma\mathcal{A}$ when $\delta g_{ij} = 2\delta\sigma g_{ij}$. On general grounds, [3, 4], $\mathcal{A} = a\,E + c\,I$ where E is the Euler density, $(R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2)/64$, and I is the square of the Weyl tensor, $I = (-R^{ijkl}R_{ijkl} + 2R^{ij}R_{ij} - R^2/3)/64$. A one-loop calculation [4] gives $\mathcal{A}$ as the sum of contributions from the six scalars, two fermions and gauge vector of the Super-Yang-Mills theory, (all in the adjoint with dimension N^2-1)

$$\mathcal{A} = \frac{(6s+2f+g_v)(N^2-1)}{16\pi^2} . \tag{1.5}$$

When the heat-kernel coefficients s, f, and g_v are expressed in terms of E and I this becomes

$$\mathcal{A} = -\frac{(N^2-1)(E+I)}{\pi^2} , \tag{1.6}$$

so $a = c = -(N^2-1)/(2\pi^2)$ and supersymmetry protects this from higher-loop corrections. Equation (1.4) shows how to find $F[g]$ in the Superstring theory. At leading order in N we can replace strings by fields, and neglect all the fields in the resulting Supergravity theory except the graviton, so S_{IIB} reduces to the Einstein-Hilbert action with cosmological term, whilst the functional integral itself can be computed in the saddle-point approximation and so reduces to the exponential of minus the action computed with the metric satsfying Einstein's equation and coinciding (up to a conformal factor) with g_{ij} when restricted to the boundary. By solving this boundary value problem in perturbation theory Henningson and Skenderis [1] were able to compute $\mathcal{A}$ to leading order in large-N.

Rather than use perturbation theory the Weyl anomaly can be calculated more simply by using an exact solution to the Einstein equations that is more general than (1.1). Replacing the Minkowski metric η_{ij} by a d-dimensional Einstein metric $\hat{g}$, ($\hat{R}_{ij} = \hat{R}\,\hat{g}_{ij}/4$ and $\hat{R} = $ constant), and modifying the warp-factor

$$ds^2 = G_{\mu\nu}dX^\mu\,dX^\nu = dr^2 + z^{-2}\,e^\rho\hat{g}_{ij}(x)dx^i dx^j , \qquad e^{\rho/2} = 1 - C\,z^2 , \qquad C = \frac{l^2\hat{R}}{4\,d(d-1)} , \tag{1.7}$$

results in a bulk metric that still satisfies (1.2) and (1.3). The metric restricted to the 'boundary' at $z=\tau$ is $\hat{g}$ up to a conformal factor. We now specialise to $d=4$. The greater generality of this metric is useful because it allows us to calculate the anomaly coefficients

a and c by making special choices for $\hat{g}$. Were we to calculate the anomaly for Ricci flat $\hat{g}$, so that $E = -I = R^{ijkl}R_{ijkl}/64$ we would find the combination $a - c$. By taking instead a $\hat{g}$ for which $\hat{R}_{ijkl} = \hat{R}(\hat{g}_{ik}\hat{g}_{jl} - \hat{g}_{jk}\hat{g}_{il})/12$ so that $I = 0$ and $E = \hat{R}^2/384$ we would obtain the coefficient a. The Einstein-Hilbert action evaluated in this metric is

$$S_{\rm EH} = \frac{1}{16\pi G_N}\int d^5X\sqrt{G}\,(R + 2\Lambda) = -\frac{1}{2\pi G_N l^2}\int \frac{dr\,d^4x}{z^4}\sqrt{\hat{g}}\,(1 - Cz^2)^4\,. \tag{1.8}$$

The integral over r diverges as the boundary is approached, hence the need for the cut-off. Because the anomaly depends on just two numbers, a and c, it is sufficient to consider a Weyl scaling of the boundary metric, $\delta g_{ij} = 2\delta\sigma g_{ij}$, with constant $\delta\sigma$, and this can be achieved by keeping $\hat{g}$ fixed, but varying r_0 by $\delta r_0 = l\delta\sigma$, so

$$\delta F \approx -\delta\sigma\, l\frac{\partial}{\partial r_0}S_{\rm EH} = -\frac{\delta\sigma}{2\pi G_N l}\int \frac{d^4x}{\tau^4}\sqrt{\hat{g}}\,(1 - C\tau^2)^4\,. \tag{1.9}$$

The divergent parts of this as $\tau \downarrow 0$ can be cancelled by adding counter-terms to the action, but the finite contribution, proportional to C^2, cannot, so we obtain the bulk tree-level contribution to the anomaly as

$$\mathcal{A}_{\rm tree} = -\frac{3l^3}{48^2\pi G_N}R^2 = -\frac{l^3}{2\pi G_N}(E + I)\,. \tag{1.10}$$

The gravitational coupling is related to N and l via $G_N = \pi l^3/(2N^2)$, [1], so this reproduces the leading term in (1.6). (Note that it is easy to check that the Gibbons-Hawking boundary action does not contribute to this calculation of $\mathcal{A}$ so we have not included it in our discussion).

To go beyond the leading order and compute the bulk one-loop contribution to the Weyl anomaly, $\delta\mathcal{A}$, requires making sense of the left-hand-side of (1.4). We will approximate the IIB Superstring theory by IIB Supergravity, which means neglecting higher orders in α'. Even so the left-hand-side is ill-defined. At the time we began our work there was no known action for this theory, but rather a set of classical equations of motion consistent with Supersymmetry which were analysed in [13] to obtain the mass spectrum for the theory compactified on $AdS_5 \times S^5$. It was not obvious that these equations of motion could be derived from an action, but in section 6 we construct one that yields the equations of motion to quadratic order in the quantum fluctuations of the fields which is what is needed to obtain the one-loop contribution to the anomaly in the bulk theory. Integrating out these fluctuations would give a functional determinant for each of the infinite number of fields in the compactification. To compute these in the conventional fashion requires the use of the heat-kernels for differential operators defined on a five-dimensional manifold with boundary, again these were unknown at the time we began. We will adopt a different approach based on the interpretation of Ψ as a wave-functional which satisfies a functional Schrödinger equation from which it can be constructed. Because this is a hamiltonian approach it involves four-dimensional differential operators whose heat-kernel coefficients are already tabulated. Furthermore, because it treats the five-dimensional bulk fields in terms of their values on the boundary it is ideally suited to discussing the Maldacena conjecture.

JHEP01(2004)013

Consider, for the sake of illustration, a scalar field of mass m. To quadratic order in the field there are no interactions other than those with the background metric, so the action is

$$\begin{aligned} S_\phi &= \frac{1}{2}\int d^5X\sqrt{G}\left(G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2\right) \\ &= \frac{1}{2}\int \frac{d^4x\,dr}{z^4}\sqrt{\hat g}\,e^{2\rho}\left(\dot\phi^2 + z^2 e^{-\rho}\hat g^{ij}\partial_i\phi\,\partial_j\phi + m^2\phi^2\right) \\ &\equiv \frac{1}{2}\int \frac{d^4x\,dr}{z^4}\sqrt{\hat g}\,e^{2\rho}\,\phi\,\Omega_s\,\phi\,, \end{aligned} \tag{1.11}$$

(here the dot denotes differentiation with respect to r) whilst the norm on fluctuations of the field, from which the functional integral volume element $\mathcal{D}\phi$ can be constructed is

$$||\delta\phi||^2 = \int d^5X\sqrt{G}\,\delta\phi^2 = \int \frac{d^4x\,dr}{z^4}\sqrt{\hat g}\,e^{2\rho}\,\delta\phi^2\,. \tag{1.12}$$

We will interpret the co-ordinate r as euclidean time, so to construct a Schrödinger equation we first re-define the field by setting $\phi = z^2\,e^{-\rho}\varphi$ to make the 'kinetic' term in the action into the standard form, and remove the explicit r-dependence from the integrand of the norm. The action becomes

$$\begin{aligned} S_\phi &= \frac{1}{2}\int d^4x\,dr\sqrt{\hat g}\left(\dot\varphi^2 + z^2 e^{-\rho}\varphi\left(\Box + \frac{\hat R}{6}\right)\varphi + \left(m^2 + \frac{4}{l^2}\right)\varphi^2\right) - \\ &\quad -\frac{1}{2}\int d^4x\,\sqrt{\hat g}\left(\dot\rho + \frac{2}{l}\right)\varphi^2\,, \\ &= S_\varphi + S_b\,, \end{aligned} \tag{1.13}$$

where $\Box$ is the 4-dimensional covariant laplacian constructed from $\hat g$. Note that $\Box + \frac{\hat R}{6}$ is the operator associated with a conformally coupled four-dimensional scalar field, its appearance should not be a surprise given that isometries of AdS act as conformal transformations on its boundary. Also note that the mass has been modified to an effective mass $\sqrt{m^2 + 4/l^2}$.

The one-loop contribution of a scalar field to the left-hand-side of (1.4) is

$$\int \mathcal{D}\phi\,e^{-S_\phi}\Big|_{\phi(r=r_0)=\hat\phi} \tag{1.14}$$

where the 'boundary' is taken at the cut-off, r_0, rather than $r = -\infty$. In terms of the 'canonical' field φ this is

$$e^{-S_b}\int \mathcal{D}\varphi\,e^{-S_\varphi}\Big|_{\varphi(r=r_0)=\hat\varphi} \equiv e^{-S_b + W[\hat\varphi, g]}\,. \tag{1.15}$$

Since the integral is gaussian, W takes the form

$$W[\hat\varphi] = F + \frac{1}{2}\int d^4x\,\sqrt{\hat g}\,\hat\varphi\Gamma\,\hat\varphi \tag{1.16}$$

where Γ is a differential operator and $F = -\frac{1}{2}\log\mathrm{Det}\,\Omega_s$ is the free-energy of the scalar field, whose variation under a Weyl transformation is the goal of our computation. As we have already observed, this functional integral can be interpreted as the vacuum wave-functional at euclidean 'time' r_0, and so it satisfies a functional Schrödinger equation that can be read-off from the action S_φ, thus if $\Psi = \exp W[\varphi]$ then

$$\frac{\partial}{\partial r_0}\Psi = -\frac{1}{2}\int d^4x\sqrt{\hat{g}}\left\{-\hat{g}^{-1}\frac{\delta^2}{\delta\varphi^2} + \tau^2 e^{-\rho}\varphi\left(\Box + \frac{\hat{R}}{6}\right)\varphi + \left(m^2 + \frac{4}{l^2}\right)\varphi^2\right\}\Psi\,. \quad (1.17)$$

So Γ satisfies

$$\frac{\partial}{\partial r_0}\Gamma = \Gamma^2 - \tau^2 e^{-\rho}\left(\Box + \frac{\hat{R}}{6}\right) - \left(m^2 + \frac{4}{l^2}\right), \quad (1.18)$$

which can be solved in powers of the differential operator by expanding

$$\Gamma = \sum_{n=0}^{\infty} b_n(r_0)\left(\Box + \hat{R}/6\right)^n . \quad (1.19)$$

This gives

$$b_0 = \pm\sqrt{m^2 + \frac{4}{l^2}} \quad (1.20)$$

in which we take the minus sign to give a normalisable wave-functional. The other coefficients in (1.19) have the property that we will make use of later of vanishing as the cut-off, r_0 is taken to $-\infty$.

The free-energy is determined in terms of the functional trace of Γ:

$$\frac{\partial}{\partial r_0}F = \frac{1}{2}\,\mathrm{Tr}\,\Gamma \quad (1.21)$$

which can be regulated using a heat-kernel

$$\mathrm{Tr}\,\Gamma = \sum_{n=0}^{\infty} b_n(r_0)\left(-\frac{\partial}{\partial s}\right)^n \mathrm{Tr}\exp\left(-s\left(\Box + \hat{R}/6\right)\right) \quad (1.22)$$

with s small. The heat-kernel has the well-known Seeley-de Witt expansion for small s

$$\mathrm{Tr}\exp\left(-s\left(\Box + \hat{R}/6\right)\right) = \int d^4x\,\sqrt{\hat{g}}\frac{1}{16\pi^2 s^2}\left(1 + s\,a_1(x) + s^2\,a_2(x) + s^3\,a_3(x) + \cdots\right) \quad (1.23)$$

with $\sqrt{\hat{g}}\,a_2 = \sqrt{g}\,8(2E - 3I)/45$. As s is made smaller and $|r_0|$ larger the only surviving contribution comes from 1, a_1 and a_2. The coefficients of 1 and a_1 diverge, but that of a_2 is finite.

As well as fixing W, these equations directly determine the Weyl anomaly.[1] As in the leading order calculation we consider a constant scaling of the boundary metric resulting

[1] In our calculation of the anomaly, to remove the cutoff dependence from the functional inner-product we imposed non-standard boundary conditions on the bulk field. The resulting wave-functional differs only by a boundary term from that obtained by the standard procedure (diagonalising the asymptotic part of the bulk field corresponding to the larger scaling dimension) and gives the same Weyl anomaly. For masses in a certain range, it is also possible to diagonalise the asymptotic corresponding to the smaller scaling dimension [11]; this gives a different result for the anomaly. For certain compactifications, for example when S^5 is replaced by $T^{1,1}$, this ambiguity becomes important, but in the present case the spectrum contains no modes with masses in the appropriate range. All of this is discussed in more detail in [8].

JHEP01(2004)013

from a shift in r_0, $\delta r_0 = l\delta\sigma$, so $\int d^4x\,\sqrt{g}\,\delta\mathcal{A} = l\frac{\partial}{\partial r_0}F = \frac{l}{2}\,\mathrm{Tr}\,\Gamma$. The divergent parts of this can be cancelled by adding counter-terms to F, but the finite contribution proportional to a_2 cannot, so we obtain the anomaly as $\delta\mathcal{A} = -\sqrt{l^2m^2+4}\,a_2/(32\pi^2)$. Now the mass dependence can be neatly expressed in terms of the scaling dimension of the field restricted to the boundary, Δ, because $\sqrt{l^2m^2+4} = \Delta - 2$ so we arrive at

$$\delta\mathcal{A} = -\frac{\Delta-2}{32\pi^2}\,a_2\,. \tag{1.24}$$

Although we have derived this formula for a scalar field it applies to all the species of fields in IIB Supergravity. To see this requires decomposing the appropriate action into 'canonical' fields so as to identify the appropriate four-dimensional operators and effective masses. We will describe the details of this in the subsequent sections of this paper, but the upshot of this decomposition of five-dimensional fields into four-dimensional variables is to introduce into the four-dimensional operators precisely those couplings to $\hat{R}$ that render them conformally covariant. Thus a_2 for a five-dimensional gauge field is the combination of heat-kernel coefficients for the operator associated with a four-dimensional gauge field, just as that for a minimally coupled five-dimensional scalar is associated with a conformally coupled four-dimensional scalar.

The scaling dimensions Δ are related to the bulk masses which were originally worked out in [13]. In table 1 we display the corresponding values of $\Delta - 2$. The multiplets are labelled by an integer $p \geq 2$, and the fields form representations of $\mathrm{SU}(4) \sim \mathrm{SO}(6)$. The four-dimensional heat-kernel coefficients have also been known for a long time and we use the values given by [14, 15]. In table 2 we list these for the cases of a Ricci flat boundary and for a boundary of constant $\hat{R}$.

If we denote the values of a_2 for the fields ϕ, ψ, A_μ, $A_{\mu\nu}$, ψ_μ, $h_{\mu\nu}$ by s, f, v, a, r, and g respectively then the contribution from a generic ($p \geq 4$) multiplet is

$$\begin{aligned}\Big(\sum(\Delta-2)a_2\Big)_{p\geq 4} &= (-4s+4a+r+f+2v)\frac{p}{3} - \\ &\quad -(105s+g+26a+8r+72f+48v)\frac{p^3}{12} + \\ &\quad +(16v+20f+10a+4r+25s+g)\frac{p^5}{12}\end{aligned} \tag{1.25}$$

whilst for the $p = 3$ multiplet it is

$$\Big(\sum(\Delta-2)a_2\Big)_{p=3} = 244f + 18g + 266s + 218v + 148a + 64r\,. \tag{1.26}$$

The $p = 2$ multiplet contains gauge fields requiring the introduction of Faddeev-Popov ghosts. Their parameters are given in table 3 along with the decomposition of the five-dimensional components of fields into four-dimensional pieces.

$$12v - 30s + 6r - 10f + 2g \tag{1.27}$$

and if we include the scalars, spinors and antisymmetric tensors the total contribution of the $p = 2$ multiplet is

$$\Big(\sum(\Delta-2)a_2\Big)_{p=2} = 12v - 6s + 6r + 6f + 2g + 12a\,. \tag{1.28}$$

Field	SO(4) rep$^{\rm n}$	SU(4) rep$^{\rm n}$	$\Delta - 2$
$\phi^{(1)}$	$(0,0)$	$(0,p,0)$	$p-2, \quad p \geq 2$
$\psi^{(1)}$	$(\frac{1}{2},0)$	$(0,p-1,1)_c$	$p-3/2, \quad p \geq 2$
$A^{(1)}_{\mu\nu}$	$(1,0)$	$(0,p-1,0)_c$	$p-1, \quad p \geq 2$
$\phi^{(2)}$	$(0,0)$	$(0,p-2,2)_c$	$p-1, \quad p \geq 2$
$\phi^{(3)}$	$(0,0)$	$(0,p-2,0)_c$	$p, \quad p \geq 2$
$\psi^{(2)}$	$(\frac{1}{2},0)$	$(0,p-2,1)_c$	$p-1/2, \quad p \geq 2$
$A^{(1)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-2,1)$	$p-1, \quad p \geq 2$
$\psi^{(1)}_{\mu}$	$(1,\frac{1}{2})$	$(1,p-2,0)_c$	$p-1/2, \quad p \geq 2$
$h_{\mu\nu}$	$(1,1)$	$(0,p-2,0)$	$p, \quad p \geq 2$
$\psi^{(3)}$	$(\frac{1}{2},0)$	$(2,p-3,1)_c$	$p-1/2, \quad p \geq 3$
$\psi^{(4)}$	$(\frac{1}{2},0)$	$(0,p-3,1)_c$	$p+1/2, \quad p \geq 3$
$A^{(2)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-3,1)_c$	$p, \quad p \geq 3$
$A^{(2)}_{\mu\nu}$	$(1,0)$	$(2,p-3,0)_c$	$p, \quad p \geq 3$
$A^{(3)}_{\mu\nu}$	$(1,0)$	$(0,p-3,0)_c$	$p+1, \quad p \geq 3$
$\psi^{(2)}_{\mu}$	$(1,\frac{1}{2})$	$(1,p-3,0)_c$	$p+1/2, \quad p \geq 3$
$\phi^{(4)}$	$(0,0)$	$(2,p-4,2)$	$p, \quad p \geq 4$
$\phi^{(5)}$	$(0,0)$	$(0,p-4,2)_c$	$p+1, \quad p \geq 4$
$\phi^{(6)}$	$(0,0)$	$(0,p-4,0)$	$p+2, \quad p \geq 4$
$\psi^{(5)}$	$(\frac{1}{2},0)$	$(2,p-4,1)_c$	$p+1/2, \quad p \geq 4$
$\psi^{(6)}$	$(\frac{1}{2},0)$	$(0,p-4,1)_c$	$p+3/2, \quad p \geq 4$
$A^{(3)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-4,1)$	$p+1, \quad p \geq 4$

Table 1: Mass spectrum. The supermultiplets (irreps of $\mathrm{U}(2,2/4)$) are labelled by the integer p. Note that the doubleton ($p=1$) does not appear in the spectrum. The (a,b,c) representation of SU(4) has dimension $(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)/12$, and a subscript c indicates that the representation is complex. (Spinors are four component Dirac spinors in AdS_5).

Field	$R_{ij}=0$: $180a_2/R_{ijkl}R^{ijkl}$	Constant R: $180a_2/R^2$
ϕ	1	-1/12
ψ	7/2	-11/12
A_μ	-11	29/3
$A_{\mu\nu}$	33	19/4
ψ_μ	-219/2	-61/4
$h_{\mu\nu}$	189	747/4

Table 2: Anomaly coefficients of massive fields on AdS_5. Note that the massive vector coefficient is $v_0+2s-2s_0$ where v_0, s, s_0 are respectively, the coefficients for the 4d gauge-fixed Maxwell operator, a conformally coupled scalar, and a minimally coupled scalar.

Substituting the values of the heat kernel coefficients for a Ricci flat boundary shows that the contribution of each supermultiplet vanishes implying that $a=c$ [5]. However if we do not specialise to this case we have to deal with the sum over multiplets labelled by p. We

Original field	Gauge fixed fields	$\Delta - 2$	$R_{ij} = 0$: $180a_2/R_{ijkl}R^{ijkl}$	Constant R: $180a_2/R^2$
A_μ	A_i	1	-11	29/3
(**15** of SU(4))	A_0	2	1	-1/12
	b_{FP}, c_{FP}	2	-1	1/12
ψ_μ	ψ_i^{irr}	3/2	-219/2	-61/4
	$\gamma^i\psi_i$	5/2	7/2	-11/12
(**4** of SU(4))	ψ_0	5/2	7/2	-11/12
	λ_{FP}, ρ_{FP}	5/2	-7/2	11/12
	σ_{GF}	5/2	-7/2	11/12
$h_{\mu\nu}$	h_{ij}^{irr}	2	189	727/4
(SU(4) singlet)	h_{0i}	3	-11	29/3
	h_{00}, h_μ^μ	$\sqrt{12}$	1	-1/12
	B_0^{FP},C_0^{FP}	$\sqrt{12}$	-1	1/12
	B_i^{FP},C_i^{FP}	3	11	-29/3

Table 3: Decomposition of gauge fields for the massless multiplet.

will evaluate this divergent sum by weighting the contribution of each supermultiplet by z^p. The sum can be performed for $|z| < 1$, and we take the result to be a regularisation of the weighted sum for all values of z. Multiplying this by $1/(z-1)$ and integrating around the pole at $z = 1$ gives a regularisation of the original divergent sum.[2] In this way all the members of any given supermultiplet are treated on an equal footing. This yields

$$\sum(\Delta - 2)a_2 = 8s + 4f + 2v \tag{1.29}$$

which remarkably depends only on the heat-kernel coefficients of fields in the Super-Yang-Mills theory. In the next section we will see that decomposing a five-dimensional vector into longitudinal and transverse pieces and solving the Schrödinger equation for them relates the heat-kernel coefficient for a vector field, v, to that for a four-dimensional gauge-fixed Maxwell field, v_0, by $v = v_0 + 2s - 2s_0$. s_0 is the coefficient for a minimally coupled four-dimensional scalar (Faddeev-Popov ghost), showing that $v - 2s = v_0 - 2s_0 = g_v$ [16]. Therefore we finally arrive at the one-loop contribution to the Weyl anomaly

$$\delta\mathcal{A} = -\sum \frac{(\Delta - 2)a_2}{32\pi^2} = -\frac{6s + 2f + g_v}{16\pi^2} \tag{1.30}$$

which is precisely what is needed to reproduce the subleading term in the exact Weyl anomaly of Super-Yang-Mills theory and verify the Maldacena conjecture.

A final point concerns the finiteness of the boundary theory. The divergence of the coefficients a_0 and a_1 in (1.23) renormalises the boundary cosmological and Newton constants, respectively, but we would expect these renormalisations to disappear in the full

[2] This regularisation is equivalent to simply taking a cut-off $p = \Lambda$ in the summation of supermultiplets with $p \geq 4$ and removing Λ dependent divergent terms at $\Lambda \to \infty$ from the regularised sum. Both these regularisations preserve supersymmetry, as they must.

theory. If we wrote down (1.23) in some superfield formalism, we would have to take the same proper-time separation for fields of different spin. So it makes sense to sum the contributions of all Supergravity fields to these coefficients. If we do so, the total a_0 contribution cancels by virtue of the equal number of bosonic and fermionic modes. The total a_1 contribution also cancels, but only after we apply the same regularisation that we used to sum the a_2 coefficients. So we find as expected that there is no overall renormalisation of the boundary Newton or cosmological constants [9].

2. Weyl anomaly for fermions

The euclidean action for a spin-1/2 fermion in the metric (1.7) is

$$\int d^{d+1}x\sqrt{G}\bar{\psi}(\gamma^\mu D_\mu - m)\psi\,. \tag{2.1}$$

The spin-covariant derivative is defined via the funfbein

$$V_0^\alpha = \frac{1}{z}\delta_0^\alpha\,, \qquad V_i^\alpha = \frac{1}{z}e^{\rho/2}\tilde{V}_i^\alpha\,, \tag{2.2}$$

where $\tilde{V}_i^\alpha$ is the vierbein for the boundary metric. Making the change of variables $\psi = z^2 e^{-\rho}\tilde{\psi}$ causes the volume element in the path-integral to become the usual flat-space one, and the kinetic term in the action acquires the usual form. The action can be written

$$\int d^{d+1}x\bar{\tilde{\psi}}\left(\gamma^0\partial_0 + ze^{-\rho/2}\gamma^i\tilde{D}_i - m\right)\tilde{\psi}\,. \tag{2.3}$$

The D_i derivative is spin-covariant with respect to the boundary metric.

We impose the following boundary conditions on $\tilde{\psi}$:

$$Q_+\tilde{\psi}(0,x) = u(x) = Q_+u(x), \qquad \tilde{\psi}^\dagger(0,x)Q_- = u^\dagger(x) = u^\dagger(x)Q_-\,, \tag{2.4}$$

for some local projection operators $Q_\pm$. The remaining projections are represented by functional differentiation. The partition function takes the form

$$\Psi[u,u^\dagger] = \exp[f + u^\dagger\Gamma u]\,, \tag{2.5}$$

and the Schrödinger equation that it satisfies can be written

$$\frac{\partial}{\partial r_0}\Psi = -\int d^dx\left(u^\dagger Q_- + \frac{\delta}{\delta u}Q_+\right)h\left(Q_+u + Q_-\frac{\delta}{\delta u^\dagger}\right)\Psi\,, \tag{2.6}$$

where $h = \tau e^{-\rho/2}\gamma^0\gamma^i\tilde{D}_i - \gamma^0 m$. Assume without loss of generality that $m \geq 0$. If we make the specific choice $Q_\pm = \frac{1}{2}(1\pm\gamma^0)$, we can write (2.6) as

$$\frac{\partial}{\partial r_0}\Psi = -\left[mu^\dagger\frac{\delta}{\delta u^\dagger} - m\frac{\delta}{\delta u}u - \tau e^{-\rho/2}u^\dagger\gamma\cdot\tilde{D}u + \tau e^{-\rho/2}\frac{\delta}{\delta u}\gamma\cdot\tilde{D}\frac{\delta}{\delta u^\dagger}\right]\Psi\,. \tag{2.7}$$

Acting on (2.5) this implies that

$$\dot{\Gamma} = -2m\Gamma + \tau e^{-\rho/2}\gamma\cdot\tilde{D} - \Gamma^2\tau e^{-\rho/2}\gamma\cdot\tilde{D}\,, \tag{2.8}$$

while f satisfies

$$\dot{f} = \frac{1}{2}\,\mathrm{Tr}(-m + \Gamma\tau e^{-\rho/2}\gamma\cdot\tilde{D})\,. \tag{2.9}$$

The factor of 1/2 takes into account the fact that the trace is over constrained variables. So that we can regulate this with a heat-kernel, we expand Γ in terms of the positive-definite operator $(\gamma\cdot\tilde{D})^2$:

$$\Gamma = \gamma\cdot\tilde{D}\sum_{n=0}^{\infty} d_n(r_0)(\gamma\cdot\tilde{D})^{2n}\,. \tag{2.10}$$

Notice that the coefficients d_n *all* vanish as $r_0 \to -\infty$. The equation (2.8) is easily solved in terms of Bessel functions, but to regulate (2.9) we again use a heat-kernel expansion

$$\mathrm{Tr}(-m + \Gamma\tau\gamma\cdot\tilde{D}) = \left(\sum_{n=0}^{\infty} d_n(r_0)\left(-\frac{\partial}{\partial s}\right)^{n+1} - m\right)\mathrm{Tr}\exp\left(-s(\gamma\cdot\tilde{D})^2\right), \tag{2.11}$$

where the heat-kernel has a Seeley-de Witt expansion like (1.23). The contribution proportional to the a_2 coefficient of $(\gamma\cdot\tilde{D})^2$ is finite as $s\to 0$ and $r_0\to-\infty$ and determines the anomaly, which is therefore proportional to m. But since $m = \Delta - 2$ we have as before

$$\delta\mathcal{A} = -\frac{\Delta-2}{32\pi^2}\,a_2\,. \tag{2.12}$$

For the spin-3/2 Rarita-Schwinger field the action that we obtain by diagonalising the five-dimensional action has the same form as the spin-1/2 field, and the Schrödinger equation takes the form (2.6). Thus the above discussion is essentially unchanged, leading again to an anomaly proportional to $m = \Delta - 2$.

3. Vector fields

We now wish to demonstrate that the result (1.24) extends to higher spin fields as well. We begin by considering the decomposition of a five-dimensional vector gauge field in AdS in terms of 'canonical' fields from which we can construct a functional Schrödinger equation. The classical action for a U(1) gauge field in the metric (1.7) is

$$S_{\mathrm{gv}} = \frac{1}{2}\int d^4x\,dr\,\sqrt{\hat{g}}\Big((\dot{A}_i - \partial_i A_r)(\dot{A}_j - \partial_j A_r)\,\hat{g}^{ij}e^{\sigma} + (\partial_i A_j - \partial_j A_i)(\partial_r A_s - \partial_s A_r)\hat{g}^{ir}\hat{g}^{js}\Big), \tag{3.1}$$

where we have set $\exp\sigma = \exp(\rho)/z^2$. We choose as a gauge condition that A_r be constant, so that the Euler-Lagrange equation corresponding to varying A_r,

$$\nabla_i\,(\hat{g}^{ij}\dot{A}_j) = 0 \tag{3.2}$$

must be imposed as a constraint, with ∇ the Levi-Civita connection constructed from $\hat{g}_{ij}$. If we change variables from A to $\mathcal{A} = e^{\sigma/2}A$ the kinetic term assumes the canonical form and the action becomes

$$S_{\mathrm{gv}} = \frac{1}{2}\int d^4x\,dr\,\sqrt{\hat{g}}\left(\dot{\mathcal{A}}_i\dot{\mathcal{A}}_j\,\hat{g}^{ij} + \frac{1}{l^2}\mathcal{A}_i\mathcal{A}_j\,\hat{g}^{ij} + e^{-\sigma}(\partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i)(\partial_r\mathcal{A}_s - \partial_s\mathcal{A}_r)\,\hat{g}^{ir}\hat{g}^{js}\right), \tag{3.3}$$

whilst the general co-ordinate invariant inner product on variations of the gauge field, (from which the functional integral volume element can be constructed) also takes the form appropriate to a canonical field theory

$$|| \, \delta A \, ||^2 = \int d^4x \, dr \sqrt{\hat{g}} \, e^{\sigma} \delta A_i \delta A_j \hat{g}^{ij} = \int d^4x \, dr \sqrt{\hat{g}} \, \delta \mathcal{A}_i \delta \mathcal{A}_j \hat{g}^{ij} \,. \tag{3.4}$$

We can now write down the functional Schrödinger equation satisfied by

$$\Psi = \int \mathcal{D}A \, e^{-S} \Big|_{A(r=r_0)=\hat{A}} \,. \tag{3.5}$$

Treating r as a euclidean time and quantising using $\dot{\mathcal{A}}_i \to -(\hat{g}_{ij}/\sqrt{\hat{g}}) \, \delta/\delta\mathcal{A}_j$

$$\frac{\partial}{\partial r_0}\Psi = -\frac{1}{2}\int d^4x \, \sqrt{\hat{g}} \Big(-\frac{1}{\hat{g}}\hat{g}_{ij}\frac{\delta^2}{\delta\mathcal{A}_i\delta\mathcal{A}_j} + \frac{1}{l^2}\mathcal{A}_i\mathcal{A}_j \, \hat{g}^{ij} + \\ + e^{-\sigma}(\partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i)(\partial_r\mathcal{A}_s - \partial_s\mathcal{A}_r) \, \hat{g}^{ir}\hat{g}^{js}\Big) \, \Psi \,. \tag{3.6}$$

The constraint is imposed weakly, i.e. as a condition on Ψ:

$$\nabla^i \left(g_{ij}\frac{\delta}{\delta\mathcal{A}_j} - \frac{\dot{\sigma}}{2}\mathcal{A}_i \right) \Psi = 0 \,. \tag{3.7}$$

This can be analysed by decomposing $\mathcal{A}$ as

$$\mathcal{A} = \tilde{\mathcal{A}} + \nabla\varphi \,, \qquad \nabla_i(\hat{g}^{ij}\tilde{\mathcal{A}}_j) = 0 \,, \tag{3.8}$$

so that

$$\frac{\delta}{\delta\mathcal{A}_j} = \frac{\delta}{\delta\tilde{\mathcal{A}}_j} + \nabla^j \Box^{-1}\frac{\delta}{\delta\varphi} \,. \tag{3.9}$$

The constraint becomes

$$\left(\frac{1}{\sqrt{\hat{g}}}\frac{\delta}{\delta\varphi} + \frac{\dot{\sigma}}{2}\Box\varphi \right) \Psi = 0 \,, \tag{3.10}$$

with solution

$$\Psi = \exp\left(-\frac{\dot{\sigma}}{4}\int d^4x \, \sqrt{\hat{g}}\varphi \, \Box \, \varphi \right) \, \Psi_0[\tilde{\mathcal{A}}] \tag{3.11}$$

(3.6) now implies that Ψ_0 satisfies

$$\frac{\partial}{\partial r}\Psi_0 = -\Bigg(\frac{1}{2}\int d^4x \, dr \, \sqrt{\hat{g}} \, \times \\ \times \left(-\frac{1}{\hat{g}}\hat{g}_{ij}\frac{\delta^2}{\delta\tilde{\mathcal{A}}_i\delta\tilde{\mathcal{A}}_j} + \frac{1}{l^2}\tilde{\mathcal{A}}_i\tilde{\mathcal{A}}_j \, \hat{g}^{ij} + e^{-\sigma}\tilde{\mathcal{A}}_i\left(\Box + \frac{\hat{R}}{4}\right)\tilde{\mathcal{A}}_j\hat{g}^{ij}\right) + \frac{\dot{\sigma}}{4}\,\mathrm{Tr}\,1_s\Bigg)\Psi_0 \,. \tag{3.12}$$

Comparing this to (1.17) shows that the Weyl anomaly of the gauge field is given by an effective mass of $1/l^2$ and the heat-kernel of the operator $\Box + \frac{\hat{R}}{4}$ acting on divergenceless four-dimensional vectors. Now the projector onto divergenceless four-dimensional vectors is $\mathcal{P}^i_j = \delta^i_j + \nabla_j \Box^{-1} \nabla^i$ and $(\Box + \hat{R}/4)\nabla^j = \nabla^j \Box$ when acting on scalars, so

$$\left(\Box + \frac{\hat{R}}{4}\right)^n \mathcal{P} = \left(\Box + \frac{\hat{R}}{4}\right)^n + \nabla \Box^{n-1} \nabla \tag{3.13}$$

hence

$$\mathrm{Tr}\left(e^{-s(\Box + \hat{R}/4)} \mathcal{P}\right) = \mathrm{Tr}\, e^{-s(\Box+\hat{R}/4)} - \mathrm{Tr}\, e^{-s\Box} \tag{3.14}$$

i.e. the trace of the heat-kernel for $\Box + \hat{R}/4$ acting on divergenceless vectors is equal to the trace of the heat-kernel for $\Box + \hat{R}/4$ acting on unconstrained four-dimensional vectors minus the trace of the heat-kernel for the operator $\Box$ acting on scalars,[17]. Given the origin of the term $\frac{\dot{\sigma}}{4}\, \mathrm{Tr}\, 1_s$ we regulate it using the heat-kernel for $\Box$ acting on scalars. If we denote the a_2 Seeley-de Witt coefficients for the $\Box + \frac{\hat{R}}{4}$ acting on unconstrained four-dimensional vectors by v_0 and for $\Box$ acting on scalars by s_0, then the 1-loop contribution of the gauge-field to the boundary Weyl anomaly is $\delta\mathcal{A} = -(v_0 - 2s_0)/(32\pi^2)$. This combination of heat-kernel coefficients is precisely that that arises in the Weyl anomaly of a four-dimensional gauge-field after gauge-fixing in the Lorentz-gauge, the $2s_0$ corresponding to the Faddeev-Popov ghosts which are minimally coupled to the metric. This should not be surprising since the two pieces of this expression do not correspond to conformally invariant actions, but their sum, a four-dimensional gauge-theory, does.

The S^5 compactification of IIB Supergravity produces mass terms to be added to the action $S_{\rm gv}$ of the form

$$S_{\rm mgv} = \frac{m^2}{2}\int d^4x\, dr\, \sqrt{G}\, G^{\mu\nu} A_\mu A_\nu = \frac{m^2}{2}\int d^4x\, dr\, \sqrt{\hat{g}}\left(e^{2\sigma} A_r^2 + e^{\sigma} \hat{g}^{ij} A_i A_j\right). \tag{3.15}$$

This breaks the gauge invariance and couples A_r to the longitudinal part of A_i. To decouple these degrees of freedom we change variables to $\tilde{A}_i$, which is constrained to be divergenceless, and u and w, where

$$A_i = \tilde{A}_i + \nabla_i\left(u + \Box^{-1/2} e^{-\sigma} \partial_r\left(e^{3\sigma/2} w\right)\right), \qquad A_r = \partial_r u + \Box^{1/2} e^{-\sigma/2} w\,, \tag{3.16}$$

so that the mass term becomes

$$S_{\rm mgv} = \frac{m^2}{2}\int d^4x\, dr\, \sqrt{\hat{g}} e^{2\sigma}\left(e^{-\sigma}\hat{g}^{ij}\tilde{A}_i\tilde{A}_j + \dot{u}^2 + e^{-\sigma} u\,\Box\, u + w\,\Omega_w\, w\right), \tag{3.17}$$

where

$$\Omega_w\, w = e^{-\sigma/2}\partial_r\left(e^{-\sigma}\partial_r\left(e^{3\sigma/2} w\right)\right) - e^{-\sigma}\Box w \tag{3.18}$$

u decouples from S_{gv} which becomes

$$S_{\rm gv} = \frac{1}{2}\int d^4x\, dr\, \sqrt{\hat{g}}\left(\partial_r\tilde{A}_i\,\partial_r\tilde{A}_j\,\hat{g}^{ij} e^{\sigma} + \tilde{A}_i\left(\Box + \hat{R}/4\right)\tilde{A}_j\hat{g}^{ij} + e^{2\sigma}\, w\,\Omega_v^2\, w\right). \tag{3.19}$$

The norm on fluctuations of the field copies the form of the mass term:

$$||\,\delta A\,||^2 = \int d^4x\, dr\, \sqrt{\hat{g}} e^{2\sigma}\left(e^{-\sigma}\hat{g}^{ij}\delta\tilde{A}_i\delta\tilde{A}_j + \delta\dot{u}^2 + e^{-\sigma}\delta u\,\Box\,\delta u + \delta w\,\Omega\,\delta w\right), \tag{3.20}$$

so that the functional integration volume element factorises into

$$\mathcal{D}A = \mathcal{D}\tilde{A}\,\mathcal{D}u\,\sqrt{\mathrm{Det}\,\Omega_s}\,\mathcal{D}w\,\sqrt{\mathrm{Det}\,\Omega_w} \tag{3.21}$$

where Ω_s is the same operator that occurred earlier in the discussion of the scalar field (1.11). When m is non-zero the integral over u in $\int \mathcal{D}A\,\exp(-S_{\rm gv}-S_{\rm mgv})$ generates $1/\sqrt{\mathrm{Det}\Omega_s}$ which cancels the corresponding jacobian factor, and the integral over w generates $1/\sqrt{\mathrm{Det}\,\Omega_w(\Omega_w+m^2)}$, part of which cancels the jacobian factor for w, leaving $1/\sqrt{\mathrm{Det}\,(\Omega_w+m^2)}$. Representing this determinant as another functional integral means that we can re-write the original functional integral as

$$\begin{aligned}\int \mathcal{D}A\,e^{-S_{\rm gv}-S_{\rm mgv}} &= \int \mathcal{D}\tilde{A}\,e^{-\frac{1}{2}\int d^4x\,dr\,\sqrt{\hat{g}}\left(\partial_r\tilde{A}_i\,\partial_r\tilde{A}_j\,\hat{g}^{ij}\,e^{\sigma}+\tilde{A}_i\left(\Box+\hat{R}/4\right)\tilde{A}_j\hat{g}^{ij}\right)}\times\\ &\quad\times\int \mathcal{D}w\,e^{-\frac{1}{2}\int d^4x\,dr\,\sqrt{\hat{g}}\,e^{2\sigma}w\,(\Omega+m^2)\,w}\end{aligned} \tag{3.22}$$

After a partial integration the action in the w-integral reduces to that of a scalar field in AdS_5

$$\begin{aligned}\int d^4x\,dr\,\sqrt{\hat{g}}\,e^{2\sigma}w\,(\Omega+m^2)\,w &= \int d^4x\,dr\,\sqrt{\hat{g}}\Big(e^{-\sigma}\Big(\partial_r\left(e^{3\sigma/2}w\right)\Big)^2+\\ &\qquad\qquad + e^{\sigma}w\,\Box\,w + e^{2\sigma}m^2w^2\Big)\\ &= \int d^4x\,dr\,\sqrt{\hat{g}}\,e^{2\sigma}\Big(\dot{w}^2+\Big(m^2-\frac{3}{4}\left(\ddot{\sigma}+\dot{\sigma}^2\right)\Big)w^2+\\ &\qquad\qquad + e^{-\sigma}w\,\Box\,w\Big).\end{aligned} \tag{3.23}$$

Given that $\ddot{\sigma}+\dot{\sigma}^2 = 4/l^2$ we see that the squared mass has been shifted $m^2 \to m^2-3$. As we saw in (1.24) this contributes to the Weyl anomaly with a coefficient $\sqrt{(l^2m^2-3)+4} = \sqrt{l^2m^2+1}$. If we make a change of variables similar to that for a gauge-vector $\tilde{\mathcal{A}} = e^{\sigma/2}\tilde{A}$ to turn the 'kinetic term' into canonical form then, just as for the gauge-vector the action acquires a term $\mathcal{A}^2$ which shifts the mass $m^2 \to m^2+1/l^2$ so that the coefficient of the Weyl anomaly is again $\sqrt{l^2m^2+1}$. Thus $\delta\mathcal{A}$ for the product of functional integrals in (3.22) is $-\sqrt{l^2m^2+1}\,v/(32\pi^2)$. It remains to identify the combination of heat-kernel coefficients in v. We can do this by comparing this result with the corresponding calculation for the gauge-vector. When $m=0$ the functional integral over u does not produce $1/\sqrt{\mathrm{Det}\Omega_s}$ because the appropriate part of the action vanishes, so the jacobian factor does not cancel, but rather the integration generates the volume of gauge transformations which has to be divided out in order to restrict the integral to physical degrees of freedom. Now the uncancelled jacobian factor involves the determinant of the operator associated with a massless scalar field in AdS, so it contributes mith an 'effective mass' $\sqrt{4}$ and heat kernel coefficient s (belonging to the conformally coupled operator $\Box - \hat{R}/6$ in four-dimensions) and the 'wrong' sign, because it is a jacobian. So, computed this way the Weyl anomaly of a gauge-vector is $-(v-2s)/(32\pi^2)$, but since we have already found this to be $-(v_0-2s_0)/(32\pi^2)$ we conclude that $v = v_0 + 2s - 2s_0$, as stated in the introduction. Finally we give an alternative derivation of the above results using a five-dimensionally covariant formulation, which is useful in the study of more complicated systems like the graviton case in section 4 and the case of an anti-symmetric tensor in section 5. The five-dimensional lagrangian is

$$\mathcal{L}_A = \sqrt{g}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu\right]. \tag{3.24}$$

With $R_{\mu\nu} = -4l^{-2}g_{\mu\nu}$, the equation of motion for the massive case can be written as

$$(-\Box - 4l^{-2} + m^2)\, A_\mu = 0\,, \qquad \nabla^\mu A_\mu = 0\,, \tag{3.25}$$

suggesting that we decompose the path-integral variable as $A_\mu = \hat{A}_\mu + \partial_\mu\varphi$ with $\nabla^\mu \hat{A}_\mu = 0$,

$$Z_A = \int \mathcal{D}A\, e^{-\int \mathcal{L}_A} = \int \mathcal{D}\hat{A}\, e^{-\int\sqrt{g}[\frac{1}{4}F^2(\hat{A}) + \frac{1}{2}m^2\hat{A}^2]} \int \mathcal{D}\varphi |\triangle_s|^{\frac{1}{2}}\, e^{-\frac{1}{2}m^2\int\sqrt{g}\,\varphi\triangle_s\varphi}\,, \tag{3.26}$$

where $\triangle_s = -\Box$ acting on the scalar φ and the factor $|\triangle_s|^{\frac{1}{2}} = \mathrm{Det}(-\Box)^{\frac{1}{2}}$ arises from the jacobian. For $m^2 \neq 0$, the jacobian is suppressed by the path-integral for φ and we have

$$Z_{\text{massive}} = Z_{\hat{A}} = \int \mathcal{D}\hat{A}\, e^{-\frac{1}{2}\int\sqrt{g}\,\hat{A}_a(-\Box - 4l^{-2} + m^2)\hat{A}^a} \equiv |\triangle_{\hat{A}}^{m^2}|^{-\frac{1}{2}}\,, \tag{3.27}$$

while for $m^2 = 0$, $\int \mathcal{D}\varphi$ corresponds to the gauge volume and has to be removed from Z_A,

$$Z_{\text{massless}} = \frac{Z_A}{\int \mathcal{D}\varphi} = |\triangle_{\hat{A}}^{m^2=0}|^{-\frac{1}{2}}\, |\triangle_s|^{\frac{1}{2}}\,, \tag{3.28}$$

As shown in the above, the anomaly contribution for the massive vector is written as $-\sqrt{m^2l^2+1}\, v/32\pi^2$, while a five-dimensional minimally coupled scalar with mass m^2 gives $-\sqrt{m^2l^2+4}\, s/32\pi^2$. Therefore the anomaly contribution is

$$Z_{\text{massive}} \Rightarrow -\frac{\sqrt{m^2l^2+1}}{32\pi^2}\, v\,, \qquad Z_{\text{massless}} \Rightarrow -\frac{1}{32\pi^2}(v - 2s)\,, \tag{3.29}$$

identical to the above result obtained in the canonical formulation.

4. Graviton

For the complete proof of (1.30), we also have to investigate the graviton sector since there appears a contribution from the five-dimensional (ghost) vector field, as shown in table 3. The lagrangian for the five-dimensional graviton is obtained by expanding the Einstein-Hilbert action with cosmological constant Λ,

$$\mathcal{L}_G = \kappa^{-2}\sqrt{g}\,(-R + 2\Lambda)\,. \tag{4.1}$$

w.r.t. $h_{\mu\nu} = g_{\mu\nu} - g^{AdS}_{\mu\nu}$. The term quadratic in $h_{\mu\nu}$ becomes

$$\begin{aligned}\mathcal{L}_{G2} = \kappa^{-2}\sqrt{g}\Big[&\frac{1}{4}\tilde{h}^{\mu\nu}(-g_{\mu\lambda}g_{\nu\tau}\Box - 2R_{\mu\lambda\nu\tau} + 2R_{\mu\lambda}g_{\nu\tau})\, h^{\lambda\tau} - \frac{1}{2}\nabla^\mu\tilde{h}_{\mu\nu}\nabla^\lambda\tilde{h}_\lambda{}^\nu - \\ &- \tilde{h}^{\mu\nu}\, E_{\mu\lambda}\, h_\mu{}^\lambda - \frac{1}{2}\Lambda\,\tilde{h}^{\mu\nu}\, h_{\mu\nu}\Big]\,,\end{aligned} \tag{4.2}$$

where $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, h^\lambda{}_\lambda$ and $E_{\mu\lambda} = R_{\mu\lambda} - \frac{1}{4} R g_{\mu\lambda}$. We decompose $h_{\mu\nu}$ into their traceless and trace parts; $\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{5} g_{\mu\nu}\, h^\lambda{}_\lambda$ and $h = h^\lambda{}_\lambda$. With $\phi_{\mu\nu}$ and h, $\mathcal{L}_{G2}$ is expressed in the constant curvature background as

$$\begin{aligned}
\mathcal{L}_{G2} &= \mathcal{L}_\phi + \mathcal{L}_h + \mathcal{L}_{harm.}\ , \\
\mathcal{L}_\phi &= \frac{1}{4}\kappa^{-2}\sqrt{g}\,(\nabla^\lambda \phi^{\mu\nu} \nabla_\lambda \phi_{\mu\nu} - 2l^{-2}\,\phi^{\mu\nu}\phi_{\mu\nu})\,, \\
\mathcal{L}_h &= -\frac{3}{40}\kappa^{-2}\sqrt{g}\,(h\,(-\Box)\,h + 8l^{-2}\,h^2)\,, \\
\mathcal{L}_{harm.} &= -\frac{1}{2}\kappa^{-2}\sqrt{g}\,\nabla^\mu \tilde{h}_{\mu\nu} \nabla^\lambda \tilde{h}_\lambda{}^\nu \\
&= -\frac{1}{2}\kappa^{-2}\sqrt{g}\left(\nabla^\mu \phi_{\mu\nu} - \frac{3}{10}\nabla_\nu h\right)\left(\nabla^\lambda \phi_\lambda{}^\nu - \frac{3}{10}\nabla^\nu h\right).
\end{aligned} \tag{4.3}$$

We note that the quadratic action $S_{G2} = \int \mathcal{L}_{G2}$ is invariant under a *finite* transformation $h_{\mu\nu} = h'_{\mu\nu} + \nabla_\mu V_\nu + \nabla_\nu V_\mu$ or equivalently,

$$\phi_{\mu\nu} = \phi'_{\mu\nu} + 2\,\nabla_{(\mu} V_{\nu)}\,, \qquad h = h' + 2\,\nabla^\mu V_\mu\,, \tag{4.4}$$

where $\nabla_{(\mu} V_{\nu)}$ is the symmetric and traceless part of $\nabla_\mu V_\nu$.

Next we add a mass term to $\mathcal{L}_{G2}$,

$$\mathcal{L}_{G2} + \mathcal{L}_{m^2} \equiv \mathcal{L}_{G2} + \frac{1}{4}\kappa^{-2}\sqrt{g}\,m^2\,\tilde{h}^{\mu\nu} h_{\mu\nu} = \mathcal{L}_{G2} + \frac{1}{4}\kappa^{-2}\sqrt{g}\,m^2\left(\phi^{\mu\nu}\phi_{\mu\nu} - \frac{3}{10}\,h^2\right), \tag{4.5}$$

where the form $\tilde{h}^{ab} h_{ab}$ is required (instead of $h^{\mu\nu} h_{\mu\nu}$) to produce a mass term arising from the compactification of ten-dimensional type-IIB theory on S^5 [13].

The equation of motion for the massive graviton can be cast into the form,

$$-\Box\phi_{\mu\nu} - 2l^{-2}\phi_{\mu\nu} + m^2\phi_{\mu\nu} = 0\,, \qquad -\Box h + 8l^{-2} h + m^2 h = 0\,. \tag{4.6}$$

Decomposing $\phi_{\mu\nu} = \hat{\phi}_{\mu\nu} + 3\nabla_{(\mu}\nabla_{\nu)} h/8(m^2 + 3l^{-2})$, we can see that $\phi_{\mu\nu}$ in the first equation can be replaced with $\hat{\phi}_{\mu\nu}$ which satisfies the transversal condition $\nabla^\mu \hat{\phi}_{\mu\nu} = 0$, indicating that the partition function for the massive graviton is described by the path integral w.r.t. $\hat{\phi}_{\mu\nu}$ and h.

We decompose the path-integral variable $\phi_{\mu\nu} = \hat{\phi}_{\mu\nu} + 2\,\nabla_{(\mu} V_{\nu)}$ such that $\nabla^\mu \hat{\phi}_{\mu\nu} = 0$ and transform h as $h = \hat{h} + 2\,\nabla^\mu V_\mu$. Then using the invariance of $S_{G2}[\phi, h] = \int \mathcal{L}_{G2}$ under the gauge transformation, we see that the variable V_μ only appears in the mass term,

$$\begin{aligned}
Z_G &= \int \mathcal{D}\phi \mathcal{D}h\, e^{-\int(\mathcal{L}_{G2} + \mathcal{L}_{m^2})} \\
&= \int \mathcal{D}\hat{\phi} \mathcal{D}V \mathcal{D}\hat{h}\, |\triangle_v|^{\frac{1}{2}}\, e^{-\int \mathcal{L}_{G2}[\hat{\phi},\hat{h}] - \int \mathcal{L}_{m^2}[\hat{\phi},\hat{h},V]} \\
&= \int \mathcal{D}\hat{\phi}\, e^{-\int \mathcal{L}^{m^2}_{\hat{\phi}}} \int \mathcal{D}V \mathcal{D}\hat{h}\, |\triangle_v|^{\frac{1}{2}}\, e^{-\int \mathcal{L}^{m^2}_{\hat{h},V}}\ ,
\end{aligned} \tag{4.7}$$

where

$$\mathcal{L}^{m^2}_{\hat{\phi}} = \frac{1}{4}\kappa^{-2}\sqrt{g}\left[\nabla^\lambda \hat{\phi}^{\mu\nu} \nabla_\lambda \hat{\phi}_{\mu\nu} + (m^2 - 2l^{-2})\,\hat{\phi}^{\mu\nu}\hat{\phi}_{\mu\nu}\right], \tag{4.8}$$

$$
\begin{aligned}
\mathcal{L}^{m^2}_{\hat{h},V} &= -\frac{3}{25}\,\kappa^{-2}\sqrt{g}\left[\hat{h}\,(-\Box)\,\hat{h} + 5l^{-2}\,\hat{h}^2\right] + \\
&\quad + \frac{1}{4}\kappa^{-2}\sqrt{g}\,m^2\left[2\,V_\mu \triangle_v^{\mu\nu} V_\nu - \frac{3}{10}\,(\hat{h} + 2\nabla^\mu V_\mu)^2\right],
\end{aligned} \tag{4.9}
$$

and the operator $\triangle_v$ acting on a vector field V_ν is

$$
\triangle_{v\,\mu}{}^{\nu} V_\nu \equiv -2\,\nabla^\nu \nabla_{(\mu} V_{\nu)} = \left(-\Box_\mu{}^\nu + 4l^{-2}\delta_\mu{}^\nu - \frac{3}{5}\nabla_\mu\nabla^\nu\right) V_\nu\,. \tag{4.10}
$$

which can be factorized under the decomposition of the path integral variable $A_\mu = \hat{A}_\mu + \partial_\mu \varphi$ with $\nabla^\mu \hat{A}_\mu = 0$ as

$$
\begin{aligned}
|\triangle_v|^{-\frac{1}{2}} &= \int \mathcal{D}A\, e^{-\frac{1}{2}\int\sqrt{g}\,A_\mu \triangle_v^{\mu\nu} A_\nu} \\
&= \int \mathcal{D}\hat{A} e^{-\frac{1}{2}\int\sqrt{g}\,\hat{A}^\mu(-\Box+4l^{-2})\hat{A}_\mu} \int \mathcal{D}\varphi\,|\triangle_s|^{\frac{1}{2}}\,e^{-\frac{4}{5}\int\sqrt{g}\,\varphi(\triangle_s^2+5l^{-2}\triangle_s)\varphi} \\
&= |\triangle_{\hat{A}}^{m^2=8l^{-2}}|^{-\frac{1}{2}}\,|\triangle_s + 5l^{-2}|^{-\frac{1}{2}}\,,
\end{aligned} \tag{4.11}
$$

where $|\triangle_{\hat{A}}^{m^2}|^{-\frac{1}{2}}$ is given in (3.27).

For $m^2 \neq 0$, the path-integral for V_μ and $\hat{h}$ in (4.7) is performed in a similar way after decomposing $V_\mu = \hat{V}_\mu + \partial_\mu \varphi$ s.t. $\nabla^\mu \hat{V}_\mu = 0$,

$$
\begin{aligned}
\int \mathcal{D}V\mathcal{D}\hat{h}\,|\triangle_v|^{\frac{1}{2}}\,e^{-\int \mathcal{L}^{m^2}_{\hat{h},V}} &= |\triangle_{\hat{A}}^{m^2=8l^{-2}}|^{\frac{1}{2}}|\triangle_s + 5l^{-2}|^{\frac{1}{2}} \int \mathcal{D}\hat{V} e^{-\frac{1}{2}\kappa^{-2}m^2\int\sqrt{g}\,V_\mu(-\Box+4l^{-2})V^\mu} \times \\
&\quad \times \int \mathcal{D}\varphi\mathcal{D}\hat{h}\,|\triangle_s|^{\frac{1}{2}} \times \\
&\quad \times e^{-\frac{\beta}{4}\kappa^{-2}\int\sqrt{g}\,[\frac{8}{5}\hat{h}\triangle_s\hat{h}+(8l^{-2}+m^2)\hat{h}^2-\frac{20}{3}m^2\,\varphi(\triangle_s^2+8l^{-2}\triangle_s)\varphi-4m^2\,\hat{h}\triangle_s\varphi]} \\
&= |\triangle_s + 5l^{-2}|^{\frac{1}{2}}\,|\triangle_s|^{\frac{1}{2}} \times \\
&\quad \times \begin{vmatrix} \frac{8}{5}\triangle_s + 8l^{-2} + m^2 & -2m^2\,\triangle_s \\ -2m^2\,\triangle_s & -\frac{20}{3}m^2\,(\triangle_s^2 + 8l^{-2}\triangle_s) \end{vmatrix}^{-\frac{1}{2}} \\
&\sim |\triangle_s + 8l^{-2} + m^2|^{-\frac{1}{2}}\,,
\end{aligned} \tag{4.12}
$$

while the $\hat{\phi}$-integral in (4.7) is denoted as

$$
\int \mathcal{D}\hat{\phi}\,e^{-\int \mathcal{L}^{m^2}_{\hat{\phi}}} = \int \mathcal{D}\hat{\phi}\,e^{-\frac{1}{4}\kappa^{-2}\int\sqrt{g}\,\hat{\phi}^{\mu\nu}(-\Box-2l^{-2}+m^2)\hat{\phi}_{\mu\nu}} \equiv |\triangle_{\hat{\phi}}^{m^2}|^{-\frac{1}{2}}\,, \tag{4.13}
$$

which gives

$$
Z_G = |\triangle_{\hat{\phi}}^{m^2}|^{-\frac{1}{2}}\,|\triangle_s + 8l^{-2} + m^2|^{-\frac{1}{2}}\,. \tag{4.14}
$$

Compared with (4.6), we see that the determinant $|\triangle_s + 8l^{-2} + m^2|^{-\frac{1}{2}}$ stems from the trace part h, which however is coupled to other scalars π and b in ten dimensional type-IIB theory on S^5 and indeed is given by π as $h = (16/15)\pi$ for the massive case ($m^2 = k(k+4)l^{-2}$, $k \geq 1$) [13]. As the mass spectrum of π and b is listed in table 3 of [13] and has already been counted in our sum of KK-modes, the anomaly contribution from the massive graviton corresponds to the $\hat{\phi}$-mode, $|\triangle_{\hat{\phi}}^{m^2}|^{-\frac{1}{2}}$.

For $m^2 = 0$, we have to take into account the trace mode. From (4.7), (4.8), (4.9) with $m^2 = 0$, we have

$$\begin{aligned} Z_G &= \int \mathcal{D}\hat{\phi}\, e^{-\int \mathcal{L}^{m^2}_{\hat{\phi}}} \int \mathcal{D}\hat{h}\, e^{-\frac{2\beta}{5}\kappa^{-2}\int \sqrt{g}\,\hat{h}(-\Box+5l^{-2})\hat{h}} \int \mathcal{D}V\, |\triangle_v|^{\frac{1}{2}} \\ &= |\triangle^{m^2=0}_{\hat{\phi}}|^{-\frac{1}{2}}\, |\triangle_s + 5l^{-2}|^{-\frac{1}{2}}\, |\triangle^{m^2=8l^{-2}}_{\hat{A}}|^{\frac{1}{2}}\, |\triangle_s + 5l^{-2}|^{\frac{1}{2}} \int \mathcal{D}V \\ &= |\triangle^{m^2=0}_{\hat{\phi}}|^{-\frac{1}{2}}\, |\triangle^{m^2=8l^{-2}}_{\hat{A}}|^{\frac{1}{2}} \int \mathcal{D}V\,, \end{aligned} \tag{4.15}$$

where $\int \mathcal{D}V$ corresponds to the gauge volume of the massless graviton theory and thus has to be discarded from Z_G.

Solving the Schrödinger equation, we see that the anomaly contribution from the traceless and transversal $\hat{\phi}$-mode, $|\triangle^{m^2}_{\hat{\phi}}|^{-\frac{1}{2}}$ is $-\sqrt{m^2l^2+4}\, g/32\pi^2$ with a mass-independent parameter g, while as noted in section 2, $|\triangle^{m^2=8l^{-2}}_{\hat{A}}|^{-\frac{1}{2}}$ gives $-\sqrt{8+1}v/32\pi^2 = -3v/32\pi^2$. Therefore the anomaly contributions from the massive and massless graviton are

$$Z_G^{\text{massive}} \Rightarrow -\frac{\sqrt{m^2l^2+4}}{32\pi^2}\, g\,, \qquad Z_G^{\text{massless}} \Rightarrow -\frac{1}{32\pi^2}\,(2g-3v)\,, \tag{4.16}$$

where $v = v_0 + 2s - 2s_0$, which completes the proof of (1.30).

5. Anti-symmetric tensor

Finally, we consider the theory of a massive anti-symmetric field $B_{\mu\nu}$. The five-dimensional lagrangian is given as

$$\begin{aligned} \mathcal{L}_B &= \sqrt{g}\left[\frac{1}{24}\, F_{\mu\nu\lambda}F^{\mu\nu\lambda} + \frac{1}{2}\, m^2 B_{\mu\nu}B^{\mu\nu}\right] \\ &= \frac{1}{2}\sqrt{g}\left[(\nabla_\lambda B_{\mu\nu})^2 - 2(\nabla^\mu B_{\mu\nu})^2 + B^{\lambda\tau}(2R_\lambda{}^\mu\delta_\tau{}^\nu - R_{\lambda\tau}{}^{\mu\nu})B_{\mu\nu} + m^2 B^{\mu\nu}B_{\mu\nu}\right]. \end{aligned} \tag{5.1}$$

As in the vector case, we decompose $B_{\mu\nu}$ into the transversal part $\hat{B}_{\mu\nu}$ with $\nabla^\mu\hat{B}_{\mu\nu} = 0$ and the gauge mode $\partial_{[\mu}\zeta_{\nu]}$. Then the partition function is written as

$$Z_B = \int \mathcal{D}B\, e^{-\int \mathcal{L}_B} = \int \mathcal{D}\hat{B}\, e^{-\int \sqrt{g}[\frac{1}{24}F^2(\hat{B}) + \frac{1}{2}m^2\hat{B}^2]} \int \mathcal{D}\zeta |\triangle_\zeta|^{\frac{1}{2}}\, e^{-\frac{1}{4}m^2\int\sqrt{g}\,\zeta_\mu\triangle^{\mu\nu}_\zeta\zeta_\nu}\,, \tag{5.2}$$

where the determinant of the operator $\triangle^{\mu\nu}_\zeta = -\Box^{\mu\nu} - 4l^{-2}g^{\mu\nu} + \nabla^\mu\nabla^\nu$ is expressed under the decomposition $\zeta_\mu = \hat{\zeta}_\mu + \partial_\mu\phi$ with $\nabla^\mu\hat{\zeta}_\mu = 0$,

$$\begin{aligned} |\triangle_\zeta|^{-\frac{1}{2}} &= \int \mathcal{D}\zeta\, e^{-\frac{1}{2}\int\sqrt{g}\,\zeta_\mu\triangle^{\mu\nu}_\zeta\zeta_\nu} \\ &= \int \mathcal{D}\hat{\zeta}\, e^{-\frac{1}{2}\int\sqrt{g}\,\hat{\zeta}_\mu\triangle^{\mu\nu}_\zeta\hat{\zeta}_\nu} \int \mathcal{D}\phi\, |\triangle_s|^{\frac{1}{2}} \\ &= |\triangle^{m^2=0}_{\hat{A}}|^{-\frac{1}{2}}\, |\triangle_s|^{\frac{1}{2}} \int \mathcal{D}\phi\,, \end{aligned} \tag{5.3}$$

where $|\triangle_{\hat{A}}^{m^2=0}|^{-\frac{1}{2}}$ is given in (3.27), showing that the determinant diverges due to the gauge invariance of $\triangle_\zeta^{\mu\nu}$ under $\delta\zeta_\mu = \partial_\mu \phi$. For $m^2 \neq 0$, however the determinant is suppressed by the path integral w.r.t. ζ and the remaining part in (5.2) contributes to the Weyl anomaly as

$$Z_B^{\text{massive}} = \int \mathcal{D}\hat{B}\, e^{-\int \sqrt{g}[\frac{1}{24}F^2(\hat{B})+\frac{1}{2}m^2\hat{B}^2]} \Rightarrow -\frac{|m|}{32\pi^2}\, b\,, \tag{5.4}$$

where b is a mass independent parameter. For $m^2 = 0$, the path integral for $\hat{B}_{\mu\nu}$ gives no effect to the anomaly, although there remains the jacobian in (5.2) giving

$$\int \mathcal{D}\zeta\, |\triangle_\zeta|^{\frac{1}{2}} = |\triangle_{\hat{A}}^{m^2=0}|^{\frac{1}{2}}\, |\triangle_s|^{-\frac{1}{2}} \left(\int \mathcal{D}\phi\right)^{-1} \int \mathcal{D}\hat{\zeta}\, \mathcal{D}\phi\, |\triangle_s|^{\frac{1}{2}} = |\triangle_{\hat{A}}^{m^2=0}|^{\frac{1}{2}} \int \mathcal{D}\hat{\zeta}\,, \tag{5.5}$$

where $\int \mathcal{D}\hat{\zeta}$ for the constrained variable $\hat{\zeta}_\mu$ with $\nabla^\mu \hat{\zeta}_\mu = 0$ corresponds to the gauge volume of the theory and has to be removed, which leads to the contribution to the anomaly

$$Z_B^{\text{massless}} = |\triangle_{\hat{A}}^{m^2=0}|^{\frac{1}{2}} \Rightarrow -\frac{1}{32\pi^2}(-v)\,. \tag{5.6}$$

Thus the vector parameter arising from the massless $B_{\mu\nu}$ is also given by $v = v_0 - 2s_0 + 2s$. Note that, however, the massless anti-symmetric tensor only appears in the doubleton supermultiplet, which consists of the first three fields with $p = 1$ in table 1, that is, $A_{\mu\nu}^{(1)}$ with $\triangle - 2 = 0$. As the doubleton is known to correspond to the center-of-mass degree of freedom of the boundary theory, that is, the U(1) factor of $\mathrm{U}(N) = \mathrm{SU}(N) \times \mathrm{U}(1)$ Super-Yang-Mills, we did not count the doubleton sector in the summation (1.30).

6. Diagonalisation of the spectrum

To construct the spectrum we reduce the ten-dimensional type-IIB Supergravity action about the $AdS_5 \times S^5$ background, expanding in S^5 spherical harmonics to obtain a five-dimensional action on AdS_5. For the purposes of calculating the one-loop Weyl anomaly, we need the quadratic part of the 5d action. At the time that we began this work, this had not been calculated,[3] so we constructed an action that reproduces the field equations of ten-dimensional type-IIB Supergravity, expanded in S^5 harmonics. The difficulties associated with a lagrangian description of self-dual field strengths can be avoided by expanding in S^5 harmonics before constructing the lagrangian: thus the action we construct is local in AdS_5 but not in the ten-dimensional space.

6.1 Graviton

To begin with we exclude couplings between the metric and gauge field, and consider the five-dimensional equations of motion that arise from pure ten-dimensional gravity on the

[3] A construction of the quadratic action has since been given in [12]. This is equivalent to the action we constructed and describe here, although the details of the derivation are different.

$AdS_5 \times S^5$ background. Decomposing the ten-dimensional metric into backround values and fluctuations as $g_{mn} = \dot{g}_{mn} + h_{mn}$, we have

$$\dot{R}_{\mu\nu} = \frac{4}{l^2} g_{\mu\nu}\,, \qquad \dot{R}_{\alpha\beta} = -\frac{4}{l^2} g_{\alpha\beta}\,, \tag{6.1}$$

where $g_{\mu\nu}$ is the deformed AdS_5 metric (1.7), and $g_{\alpha\beta}$ is the metric of S^5. Indices will be raised and lowered with these metrics.

The ten-dimensional Einstein term can be written to quadratic order as

$$\begin{aligned}\sqrt{g}R|_{\dot{g}+h} = (\sqrt{\dot{g}}\dot{R}) - h^{mn}\left(\sqrt{\dot{g}}\dot{R}_{mn} - \frac{1}{2}\sqrt{\dot{g}}\dot{g}_{mn}\dot{R}\right) - \\ -\frac{1}{2}h^{mn}\Bigg(\sqrt{\dot{g}}E_{mn}{}^{pq}h_{pq} + \frac{1}{2}\dot{g}^{pq}h_{pq}\left(\sqrt{\dot{g}}\dot{R}_{mn} - \frac{1}{2}\sqrt{\dot{g}}\dot{g}_{mn}\dot{R}\right) - \\ -\frac{1}{2}\sqrt{\dot{g}}h_{mn}\dot{R} + \frac{1}{2}\sqrt{\dot{g}}\dot{g}_{mn}h^{pq}\dot{R}_{pq} - \frac{1}{2}\sqrt{\dot{g}}\dot{g}_{mn}\dot{g}^{pq}E_{pq}{}^{rs}h_{rs}\Bigg)\,,\end{aligned} \tag{6.2}$$

where Roman indices refer to ten-dimensional coordinates, and E_{mnpq} is a second-order differential operator. Call the part quadratic in h_{mn} S_2. We will impose the gauge conditions

$$D^{\alpha}h_{\alpha\beta} = 0\,, \qquad D^{\alpha}h_{\alpha\mu} = 0\,. \tag{6.3}$$

From the ten-dimensional Einstein equations (6.1) we have

$$\begin{aligned}\dot{R} &= 0 \\ \dot{R}^{mn}h_{mn} &= \frac{4}{l^2}(h^{\mu}_{\mu} - h^{\alpha}_{\alpha})\,.\end{aligned} \tag{6.4}$$

Also,

$$\begin{aligned}-h^{mn}\dot{R}_{mrsn}h^{rs} &= \frac{1}{l^2}(h^{\mu\nu}h_{\mu\nu} - h^{\mu}_{\mu}h^{v}_{v}) + \frac{1}{l^2}(h^{\alpha\beta}h_{\alpha\beta} - h^{\alpha}_{\alpha}h^{\beta}_{\beta}) \\ h^{mn}\dot{R}^{r}_{m}h_{nr} &= \frac{4}{l^2}(h^{\mu v}h_{\mu v} - h^{\alpha\beta}h_{\alpha\beta}) \\ \dot{g}^{pq}E_{pq}{}^{rs}h_{rs} &= \frac{1}{2}(2\Box_x + \Box_y)h^{m}_{m} - D_m D_n h^{mn} \\ h^{mn}E_{mn}{}^{pq}h_{pq} &= h^{mn}\left(\frac{1}{2}(\Box_x + \Box_y) + D_m D_n h^{r}_{r} - D_m D^{r}h_{rn}\right),\end{aligned} \tag{6.5}$$

so that

$$\begin{aligned}\frac{-2S_2}{\sqrt{\dot{g}}} = \frac{1}{2}h^{mn}(\Box_x + \Box_y)h_{mn} - \frac{1}{2}h^{m}_{m}(\Box_x + \frac{1}{2}\Box_y)h^{n}_{n} + \\ +h^{mn}D_m D_n h^{r}_{r} - h^{mn}D_m D^{r}h_{rn} + \\ +\frac{5}{l^2}h^{\mu v}h_{\mu v} - \frac{5}{l^2}h^{\alpha\beta}h_{\alpha\beta} + \frac{3}{l^2}h^{\mu}_{\mu}h^{v}_{v} - \frac{3}{l^2}h^{\alpha}_{\alpha}h^{\beta}_{\beta}\,.\end{aligned} \tag{6.6}$$

Taking a variation of the action (6.2) with respect to h^{mn} gives the equation of motion (neglecting mass terms)

$$\begin{aligned}2\dot{R}_{mn} - \dot{g}_{mn}\dot{R} = (\Box_x + \Box_y)h_{mn} - \dot{g}_{mn}(\Box_x + \frac{1}{2}\Box_y)h^{p}_{p} + D_{(m}D_{n)}h^{p}_{p} + \\ +\dot{g}_{mn}D^{p}D^{q}h_{pq} - 2D_{(m}D^{r}h_{n)r}\,.\end{aligned} \tag{6.7}$$

This contracts to

$$\dot{R} = (\Box_x + \frac{1}{2}\Box_y)h_p^p - D^p D^q h_{pq}\,, \tag{6.8}$$

whence

$$\begin{aligned} 2\dot{R}_{(\mu\nu)} &= (\Box_x + \Box_y)h'_{\mu\nu} + D_{(\mu}D_{\nu)}h'^{\mu}_{\mu} - 2D_{(\mu}D^{\rho}h'_{\nu)\rho} \\ &= 2E^{1.1}_{(\mu\nu)}\,. \end{aligned} \tag{6.9}$$

Here $h'_{\mu\nu}$ is defined by a linearised Weyl shift: $h_{\mu\nu} = h'_{\mu\nu} - \frac{1}{3}g_{\mu\nu}h^{\alpha}_{\alpha}$. Round brackets indicate that an index pair is symmetrised with the trace removed. In writing (6.9) we made use of the gauge conditions (6.3). Also, we have

$$\begin{aligned} 2g^{\mu\nu}R_{\mu\nu} &= (2\Box_x + \Box_y)h^{\alpha}_{\alpha} - \frac{5}{3}(\Box_x + \Box_y)h'^{\mu}_{\mu} - 2D^{\mu}D^{\nu}h'_{\mu\nu} \\ &= 10E^{1.2}\,, \end{aligned} \tag{6.10}$$

$$\begin{aligned} 2R_{(\alpha\beta)} &= (\Box_x + \Box_y)h_{(\alpha\beta)} + (D_{(\alpha}D_{\beta)}h^{\alpha}_{\alpha} - \frac{16}{15}D_{(\alpha}D_{\beta)}h'^{\mu}_{\mu}) - 2D_{(\alpha}D^{\mu}h_{\beta)\mu} \\ &= 2E^{3.1}_{\alpha\beta} + 2E^{3.2}_{\alpha\beta} - 2E^{3.3}_{\alpha\beta}\,, \end{aligned} \tag{6.11}$$

$$2g^{\alpha\beta}R_{\alpha\beta} = \left(\Box_x - \frac{1}{15}\Box_y\right)h'^{\mu}_{\mu} + \Box_y h^{\alpha}_{\alpha} \tag{6.12}$$

$$= 10E^{3.4}\,, \tag{6.13}$$

$$\begin{aligned} 2R_{\mu\alpha} &= \left(\delta^{\nu}_{\mu}(\Box_x + \Box_y) - D_{\mu}D^{\nu}\right)h_{\nu\alpha} + D_{\alpha}\left(D_{\mu}\left(h^{\alpha}_{\alpha} + \frac{8}{15}h'^{\mu}_{\mu}\right) - D^{\nu}h'_{\mu\nu}\right) \\ &= 2E^{2.1}_{\mu\alpha} + 2E^{2.2}_{\alpha\mu}\,. \end{aligned} \tag{6.14}$$

The equations of motion arising from the action S_2 imply the vanishing of the quantities (6.9)-(6.14). We expand everything in S^5 spherical harmonics as follows:

$$\begin{aligned} h'_{\mu\nu} &= \sum H_{\mu\nu}(x)Y(y)\,, \qquad & h_{\mu\alpha} &= \sum B_{\mu}(x)Y_{\alpha}(y)\,, \\ h_{(\alpha\beta)} &= \sum \phi(x)Y_{(\alpha\beta)}(y)\,, \qquad & h^{\alpha}_{\alpha} &= \sum \pi(x)Y(y)\,. \end{aligned} \tag{6.15}$$

When we include the couplings to the antisymmetric field, the equations of motion (6.9)–(6.14) give the equations of motion E1.1-E3.4 in table 2 of [13], each being proportional to a single spherical harmonic. We will refer throughout to the equations in table 2 of ([13]) since they give a convenient way of checking the coefficients of coupling and mass terms.

6.2 Antisymmetric tensor

Again we begin by excluding couplings to the metric, and seek to construct an action that reproduces the equations of motion in table 2 of [13]. We decompose the ten-dimensional four-index antisymmetric tensor in terms of background values and fluctuations as $A_{mnpq} = \dot{A}_{mnpq} + a_{mnpq}$ and expand the fluctuations a_{mnpq} in spherical harmonics as follows:

$$\begin{aligned} a_{\mu\nu\rho\sigma} &= \sum b_{\mu\nu\rho\sigma}(x)Y(y)\,, \\ a_{\mu\nu\rho\alpha} &= \sum b_{\mu\nu\rho}(x)Y_{\alpha}(y)\,, \end{aligned}$$

$$a_{\mu\nu\alpha\beta} = \sum b_{\mu\nu}(x) Y_{[\alpha\beta]}(y)\,,$$
$$a_{\mu\alpha\beta\gamma} = \sum \phi_\mu(x) \epsilon^{\delta\epsilon}_{\alpha\beta\gamma} D_\delta Y_\epsilon(y)\,,$$
$$a_{\alpha\beta\gamma\delta} = \sum b(x) \epsilon^{\epsilon}_{\alpha\beta\gamma\delta} D_\epsilon Y(y)\,. \tag{6.16}$$

Consider the action

$$S^{40} = b\epsilon^{\mu\nu\rho\sigma\tau}\partial_\mu b_{\nu\rho\sigma\tau} + 12b\Box_y b - \frac{1}{2} b^{\mu\nu\rho\sigma} b_{\mu\nu\rho\sigma}\,. \tag{6.17}$$

Varying b gives M1:

$$5\partial_{[\mu} b_{\nu\rho\sigma\tau]} - \epsilon_{\mu\nu\rho\sigma\tau}\Box_y b\,. \tag{6.18}$$

Varying $b^{\mu\nu\rho\sigma}$ gives M2.2:

$$-\epsilon^{\tau\mu\nu\rho\sigma}\partial_\tau b - b_{\mu\nu\rho\sigma}\,. \tag{6.19}$$

Now consider the action

$$S^{31} = \frac{1}{2} b^{\mu\nu\rho} b_{\mu\nu\rho} + b^{\mu\nu\rho}\epsilon^{\sigma\tau}_{\mu\nu\rho}\partial_\sigma\phi_\tau + 3\phi^\tau \triangle_y \phi_\tau\,. \tag{6.20}$$

Varying $b^{\mu\nu\rho}$ gives M3.2:

$$b_{\mu\nu\rho} + \epsilon^{\sigma\tau}_{\mu\nu\rho}\partial_\sigma\phi_\tau\,, \tag{6.21}$$

while varying ϕ^τ gives

$$-\epsilon^{\sigma\tau}_{\mu\nu\rho}\partial_\sigma b^{\mu\nu\rho} + 6\triangle_y\phi_\tau\,, \tag{6.22}$$

which is equivalent to M2.1:

$$4\partial_{[\mu} b_{\nu\rho\sigma]} + \epsilon^{\tau}_{\mu\nu\rho\sigma}\triangle_y\phi_\tau\,. \tag{6.23}$$

Finally, consider the action

$$S^{22} = b^+_{\mu\nu}\partial_\rho b^-_{\sigma\tau}\epsilon^{\mu\nu\rho\sigma\tau} - ib^{+\mu\nu}\sqrt{-\Delta_y}\, b^-_{\mu\nu}\,. \tag{6.24}$$

Varying $b^+_{\mu\nu}$ gives

$$\partial_\rho b^-_{\sigma\tau}\epsilon^{\mu\nu\rho\sigma\tau} - i\sqrt{-\Delta_y}\, b^-_{\mu\nu}\,, \tag{6.25}$$

which corresponds to M3.1.

6.3 Gravitational couplings

To generate the correct couplings to gravity in the equations of motion for the antisymmetric tensor we make the modifications

$$S^{40} \to S^{40}_{\text{int}} = S^{40} + \frac{12}{l} H b - \frac{32}{l}\pi b\,,$$
$$S^{31} \to S^{31}_{\text{int}} = S^{31} + \frac{6}{l}\phi^\tau B_\tau\,, \tag{6.26}$$

where $H = H^\mu_\mu$. The action at this stage is

$$S_2 + A_1 S^{40}_{\text{int}} + A_2 S^{31}_{\text{int}} + A_3 S^{22}_{\text{int}}\,, \tag{6.27}$$

and the normalisations can be fixed by considering the terms in the Einstein equations generated by the interactions. The contribution to $E^{1.2}$ is $-\frac{12}{l}A_1 b$. Now in equation E1.2 of [13] we find a term

$$\frac{1}{3l}\epsilon^{\mu\nu\rho\sigma\tau}\partial_\mu b_{\nu\rho\sigma\tau} = -\frac{8}{l}\left(\frac{1}{2l}H - \frac{4}{3l}\pi + \Box_y b\right), \tag{6.28}$$

where we used the equation of motion M1. So if $\frac{12}{l}A_1 = \frac{8}{l}\Box_y$, the correct coupling is generated, along with some mass terms. Note that $\Box_y$ has zero modes. This choice also generates the correct coupling in equation M3.4.

The contribution of the interaction terms to $E^{2.1}_{\mu\alpha}$ is $\frac{3}{l}A_2\phi_\mu$. In equation E2.1 we find the interaction terms

$$-\left(-\frac{2}{3l}\epsilon_\mu^{\nu\rho\sigma\tau}\partial_\nu b_{\rho\sigma\tau} + \frac{4}{l}\Delta_y\phi_\mu\right), \tag{6.29}$$

which we can rewrite with the help of M2.1 as

$$-\frac{8}{l}\Delta_y\phi_\mu + \frac{4}{l^2}B_\mu\,. \tag{6.30}$$

So we can take $A_2 = -\frac{4}{3}\Delta_y$. The normalisation of A_3 does not need to be fixed, as the action is diagonal in the field $b^{\pm}_{\mu\nu}$.

6.4 Mass terms

To calculate the mass terms we will drop the existing mass terms from S_2 and add a mass term to the action:

$$\begin{aligned} S_{\text{mass}} = &-\frac{1}{2}B_1 h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}B_2(h^{\mu\nu}g_{\mu\nu})^2 - \frac{1}{2}B_3 h^{\alpha\beta}h_{\alpha\beta} - \\ &-\frac{1}{2}B_4(h^{\alpha\beta}g_{\alpha\beta})^2 - Ch^\alpha_\alpha h^\mu_\mu - \frac{1}{2}Dh^\alpha_\mu h^\mu_\alpha\,. \end{aligned} \tag{6.31}$$

The coefficients of the mass terms are easily determined from the Einstein equations. From E3.1 we find $B_3 = -1/l^2$, while E1.1 gives $B_1 = 1/l^2$. E3.4, after a bit of calculation, leads to the equations

$$5B_2 - 3C + \frac{1}{l^2} = 0, \qquad 3B_4 - 5C + \frac{25}{l^2} = 0\,, \tag{6.32}$$

while E1.2, rewritten as before with the help of M1 to eliminate the four-index antisymmetric field, gives

$$3B_2 - 5C + \frac{7}{l^2} = 0, \qquad 5B_4 - 3C + \frac{31}{l^2} = 0\,. \tag{6.33}$$

These have the consistent solution $B_2 = 1/l^2$, $C = 2/l^2$, $B_4 = -5/l^2$. Finally, E2.1 gives $D = -6/l^2$, where this includes an extra -2 that arises because we wrote the action S_2 in terms of $\Box_y$ instead of Δ_y.

The final form of S_{mass}, is then

$$S_{\text{mass}} = \frac{1}{l^2} = \left(-\frac{1}{2}H^{(\mu\nu)}H_{(\mu\nu)} + \frac{1}{2}\phi^2 + 6\phi^\mu\phi_\mu - \frac{3}{5}H^2 + \frac{64}{15}\pi^2\right). \tag{6.34}$$

where we have written everything in terms of harmonically expanded fields, but the explicit dependence on spherical harmonics has been suppressed.

6.5 Diagonalisation

Writing the complete action so far in terms of the expanded fields gives:

$$
\begin{aligned}
S =& -\frac{1}{4}H^{(\mu\nu)}(\Box_x+\Box_y)H_{(\mu\nu)} - \frac{1}{2}D_\mu H^{(\mu\nu)}D^\rho H_{(\rho\nu)} - \frac{1}{2l^2}H^{(\mu\nu)}H_{(\mu\nu)} - \frac{1}{4}\phi(\Box_x+\Box_y)\phi + \\
&+\frac{1}{2l^2}\phi^2 - \frac{1}{2}\phi^\mu(\Box_x+\Box_y)\phi_\mu - \frac{1}{2}D_\mu\phi^\mu D_\nu\phi^\nu + \frac{6}{l^2}\phi^\mu\phi_\mu + H\left(\frac{3}{25}\Box_x+\frac{1}{5}\Box_y\right)H - \\
&-\frac{3}{5l^2}H^2 + \pi\left(\frac{2}{225}\Box_x - \frac{2}{15}\Box_y\right)\pi + \frac{64}{15l^2}\pi^2 - \frac{8}{30}H\Box_y\pi - \frac{3}{10}HD^\mu D^\nu H_{(\mu\nu)} + \\
&+\frac{2}{3}\tilde{b}\left(\epsilon^{\mu\nu\rho\sigma\tau}\partial_\mu b_{\nu\rho\sigma\tau} - 12(1-\delta_{I,0})\tilde{b} + \frac{12}{l}H - \frac{32}{l}\pi\right) - \frac{1}{3}b^{\mu\nu\rho\sigma}\Box_y b_{\mu\nu\rho\sigma} - \\
&-\frac{8}{3}\left(\frac{1}{2}b^{\mu\nu\rho}\Delta_y b_{\mu\nu\rho} + b^{\mu\nu\rho}\epsilon_{\mu\nu\rho}{}^{\sigma\tau}\partial_\sigma\Delta_y\phi_\tau - 3\Delta_y\phi^\tau\Delta_y\phi_\tau + \frac{6}{l}\Delta_y\phi^\tau B_\tau\right) + \\
&+A_3\left(b^+_{\mu\nu}\partial_\rho b^-_{\sigma\tau}\epsilon^{\mu\nu\rho\sigma\tau} - ib^{+\mu\nu}\sqrt{-\Delta_y}b^-_{\sigma\tau}\right).
\end{aligned}
\tag{6.35}
$$

We have defined $\tilde{b} = \Box_y b$ for future convenience, and the delta function $\delta_{I,0}$ is equal to 1 for the spherical harmonic for which $\Box_y$ has eigenvalue 0. In [13] the fields $b_{\mu\nu\rho}$ and $b_{\mu\nu\rho\sigma}$ were algebraically eliminated from the equations of motion. This is equivalent to shifting the fields: $b_{\mu\nu\rho} = b^q_{\mu\nu\rho} + b^c_{\mu\nu\rho}$ and $b_{\mu\nu\rho\sigma} = b^q_{\mu\nu\rho\sigma} + b^c_{\mu\nu\rho\sigma}$, where the "classical" parts satisfy equations of motion corresponding to M2.1 and M2.2:

$$
\Delta_y b^c_{\mu\nu\rho} + \epsilon_{\mu\nu\rho}{}^{\sigma\tau}\partial_\sigma\Delta_y\phi_\tau = 0\,, \tag{6.36}
$$

$$
b^c_{\mu\nu\rho\sigma} + \epsilon_{\mu\nu\rho\sigma}{}^{\tau}\partial_\tau\tilde{b} = 0\,. \tag{6.37}
$$

The quantum parts decouple and are non-dynamical, while the part of the action involving ϕ^τ and $\tilde{b}$ becomes

$$
8\left(-2\partial_{[\sigma}\phi_{\tau]}\Delta_y\partial^{[\sigma}\phi^{\tau]} + \Delta_y\phi^\tau\Delta_y\phi_\tau - \frac{2}{l}\Delta_y\phi^\tau B_\tau\right) - 8\tilde{b}\left((-\Box_x\Box_y^{-1}-1)\tilde{b} - \frac{1}{l}H + \frac{8}{3l}\pi\right). \tag{6.38}
$$

In this expression we have assumed for the moment that the eigenvalue of $\Box_y$ is non-zero. Acting on ϕ_μ and B_μ, Δ_y has the eigenvalues $-\frac{1}{l^2}(k+1)(k+3)$, where $k = 1,2,3,\ldots$. Also, $\Box_y = \Delta_y + \frac{4}{l^2}$. We can diagonalise the ϕ_μ, B_μ system by putting

$$
A^{(1)}_\mu = B_\mu - \frac{4}{l}(k+3)\phi_\mu\,, \qquad A^{(2)}_\mu = B_\mu - \frac{4}{l}(k+1)\phi_\mu\,, \tag{6.39}
$$

and the masses take the expected values $M^2l^2 = (k^2-1)$ and $M^2l^2 = (k+3)(k+5)$ respectively.

To diagonalise the graviton, we make the orthogonal decomposition

$$
H_{(\mu\nu)} = \hat{h}_{\mu\nu} + D_{(\mu}\Lambda_{\nu)} + D_{(\mu}D_{\nu)}\Box_x^{-1}\tilde{\phi}\,, \tag{6.40}
$$

where the components satisfy the transversality conditions $D^\mu\hat{h}_{\mu\nu} = D^\mu\Lambda_\mu = 0$. Inserting this decomposition into (6.35) there are no cross-terms. The $\tilde{\phi}$ part of the action becomes

$$
\tilde{\phi}\left(\frac{3}{25}\Box_x + \left(-\frac{3}{5l^2} - \frac{1}{5}\Box_y\right) + \frac{1}{l^2}\Box_x^{-1}\Box_y\right)\tilde{\phi} - \frac{6}{5}H\left(\frac{1}{5}\Box_x - \frac{1}{l^2}\right)\tilde{\phi}\,. \tag{6.41}
$$

To get rid of the $\Box_x^{-1}$ dependence we introduce an additional field ψ. (6.41) becomes

$$\tilde{\phi}\left(\frac{3}{25}\Box_x - \frac{3}{5l^2} - \frac{1}{5}\Box_y\right)\tilde{\phi} + 2\tilde{\phi}\psi - l^2\psi\Box_x\Box_y^{-1}\psi - \frac{6}{5}H\left(\frac{1}{5}\Box_x - \frac{1}{l^2}\right)\tilde{\phi}\,. \tag{6.42}$$

The rest of the scalar part of the action is

$$\begin{aligned} &H\left(\frac{3}{25}\Box_x - \frac{3}{5l^2} + \frac{1}{5}\Box_y\right)H + \pi\left(\frac{2}{225}\Box_y - \frac{2}{15}\Box_x + \frac{64}{15l^2}\right)\pi - \\ &-\frac{8}{30}H\Box_y\pi - 8\tilde{b}\left((-\Box_y^{-1}\Box_x - 1)\tilde{b} - \frac{1}{l}H + \frac{8}{3l}\pi\right). \end{aligned} \tag{6.43}$$

Making the change of variables $H = \phi_1 + \phi_2$, $\tilde{\phi} = \phi_1 - \phi_2$, the total scalar action (6.43)–(6.42) can be written

$$\begin{aligned} S_{\text{scalar}} = {}&\phi_1\left(\frac{4}{5}\Box_y\phi_2 + 2\psi + 8\tilde{b} - \frac{4}{15}\Box_y\pi\right) + \phi_2\left(\frac{12}{25}\Box_x - \frac{12}{5l^2}\right)\phi_2 - \\ &-\frac{1}{l^2}\psi\Box_x\Box_y^{-1}\psi + \pi\left(\frac{2}{225}\Box_y - \frac{2}{15}\Box_x + \frac{64}{15l^2}\right)\pi + 8\tilde{b}(\Box_x\Box_y^{-1})\tilde{b} + \\ &+\phi_2\left(-\frac{4}{15}\Box_y\pi + \frac{8}{l}\tilde{b} - 2\psi\right) - \frac{64}{3l}\tilde{b}\pi\,. \end{aligned} \tag{6.44}$$

The integral over ϕ_1 now imposes the condition

$$\frac{4}{5}\Box_y\phi_2 + 2\psi + 8\tilde{b} - \frac{4}{15}\Box_y\pi = 0\,, \tag{6.45}$$

which can be used to eliminate the field ψ.

Changing variables from $(\phi_2, \pi, \tilde{b})$ to

$$(X,Y,Z) = \left(\frac{\frac{5}{2}\left(\phi_2 - \frac{1}{3}\pi\right)}{\sqrt{5 + l^2\Box_y}}, i\frac{5}{\sqrt{2}}\left(\frac{5}{3}\pi - 2\phi_2\right), i\sqrt{\frac{\Box_y}{8}}\left(\tilde{b} + \frac{l^2}{5}\Box_y\phi_2\right)\right), \tag{6.46}$$

we find that the kinetic term is diagonal in (X,Y,Z). Diagonalising the mass matrix then gives mass eigenvalues $M^2l^2 = k(k-4)$, $(k+4)(k+8)$, and 5. All this assumed that the eigenvalue of $\Box_y$ was non-zero, so $k = 2, 3, \ldots$. The modes on which $\Box_y = 0$ will be considered shortly.

The part of the action involving Λ_μ can be written as

$$\frac{1}{8}\Box_y\Lambda\left(\Box_x - \frac{4}{l^2}\right)\Lambda, \tag{6.47}$$

and on spherical harmonics for which $\Box_y \neq 0$ the integration over Λ just cancels the jacobian for the change of variables (6.40). For the mode with $\Box_y = 0$ the jacobian is not cancelled. In either case, if we put together the actions for $h_{(\mu\nu)}$ Λ_μ and the scalar of mass 5, we get the correct quadratic action for a five-dimensional massive (or massless) graviton, as considered in section 4.

Finally we consider the modes for which the eigenvalue of $\Box_y$ vanishes. In this case, after the decomposition (6.40), the scalar part of the action becomes

$$\frac{3}{25}(H-\Box_x\tilde{\phi})\left(\Box_x-\frac{5}{l^2}\right)(H-\Box_x\tilde{\phi})-\frac{2}{15}\pi\left(\Box_x-\frac{32}{l^2}\right)\pi+\frac{2}{3}\tilde{b}\epsilon^{\mu\nu\rho\sigma\tau}\partial_\mu\hat{b}_{\nu\rho\sigma\tau}\,, \qquad (6.48)$$

where $\hat{b}_{\nu\rho\sigma\tau}$ has been shifted to decouple π and H from $\tilde{b}$. As a result of this shift, the action for $\tilde{b}$ corresponds to a scalar of mass $M^2l^2=45$, and is identified with the $k=1$ mode in the second branch of mass eigenvalues.

6.6 Antisymmetric tensor spectrum

The action for a free massless complex ten-dimensional antisymmetric tensor can be written as

$$S=\int d^{10}x\sqrt{-g}\left(-\frac{1}{2}\nabla_m\bar{A}_{nk}(\nabla^mA^{nk}-\nabla^nA^{mk}-\nabla^kA^{nm})\right). \qquad (6.49)$$

Writing this in terms of five-dimensional components and making use of the gauge conditions $\nabla^\alpha A_{\alpha\beta}=\nabla^\alpha A_{\alpha b}=0$ gives

$$\begin{aligned}S=\int d^{10}x\sqrt{-g}\Big(&-\frac{1}{2}\left(\nabla_\mu\bar{A}_{\alpha\beta}\nabla^\mu A^{\alpha\beta}+\nabla_\gamma\bar{A}_{\alpha\beta}\nabla^\gamma A^{\alpha\beta}+6\bar{A}_{\alpha\beta}A^{\alpha\beta}\right)-\\&-\left(\nabla_\mu\bar{A}_{\nu\alpha}(\nabla^\mu A^{\nu\alpha}-\nabla^\nu A^{\mu\alpha})+\nabla_\beta\bar{A}_{\mu\alpha}\nabla^\beta A^{\mu\alpha}+4\bar{A}_{\mu\alpha}A^{\mu\alpha}\right)-\\&-\frac{1}{2}\left(\nabla_\mu\bar{A}_{\nu\rho}(\nabla^\mu A^{\nu\rho}-\nabla^\nu A^{\mu\rho}-\nabla^\rho A^{\nu\mu})+\nabla_\alpha\bar{A}_{\mu\nu}\nabla^\alpha A^{\mu\nu}\right)\Big). \qquad (6.50)\end{aligned}$$

The three lines of (6.50) correspond to scalar, vector, and antisymmetric tensor fields on AdS_5, as is clear when we perform the expansion in spherical harmonics:

$$\begin{aligned}A_{\mu\nu}&=\sum a_{\mu\nu}(x)Y(y)\,, \qquad & A_{\mu\alpha}&=\sum a_\mu(x)Y_\alpha(y)\,,\\ A_{\alpha\beta}&=\sum a(x)Y_{\alpha\beta}(y)\,, & B&=\sum B(x)Y(y)\,. \qquad (6.51)\end{aligned}$$

To reproduce the equations of motion in [13] we must add to (6.50) a topological mass term

$$S_{\rm mass}=i\epsilon^{\alpha\beta\gamma\delta\epsilon}\bar{A}_{\alpha\beta}\varphi_\gamma A_{\delta\epsilon}+i\epsilon^{\mu\nu\rho\sigma\tau}\bar{A}_{\mu\nu}\varphi_\rho A_{\sigma\tau}\,. \qquad (6.52)$$

We can rewrite the antisymmetric tensor part of (6.50) as a first-order system by introducing auxiliary fields B_{ab} and $\bar{B}_{ab}$:

$$\begin{aligned}S_A=\int d^{10}x\sqrt{-g}\Big(&-\frac{i}{2}\epsilon^{\mu\nu\rho\sigma\tau}\bar{B}_{\mu\nu}\varphi_\rho A_{\sigma\tau}+\frac{i}{2}\epsilon^{\mu\nu\rho\sigma\tau}B_{\mu\nu}\varphi_\rho\bar{A}_{\sigma\tau}-2\bar{B}_{\mu\nu}B^{\mu\nu}-\\&-\frac{1}{2}\nabla_\alpha\bar{A}_{\mu\nu}\nabla^\alpha A^{\mu\nu}+i\epsilon^{\mu\nu\rho\sigma\tau}\bar{A}_{\mu\nu}\varphi_\rho A_{\sigma\tau}\Big). \qquad (6.53)\end{aligned}$$

Changing the variables to

$$C=\frac{1}{2}\Box_y^{1/2}A+\Box_y^{-1/2}(B-A)\,, \qquad \bar{C}=\frac{1}{2}\Box_y^{1/2}\bar{A}-\Box_y^{-1/2}(\bar{B}-\bar{A})\,, \qquad (6.54)$$

gives standard mass terms with eigenvalues as in [13]. From the point of view of calculating the anomaly, it is clear that a complex antisymmetric tensor in the first-order formalism is equivalent to a real antisymmetric tensor in the second-order formalism, as considered in section 5.

7. Fermion spectrum

The action for the ten-dimensional spinor field is

$$S = \int d^{10}x\sqrt{-g}\left(\bar{\hat{\lambda}}\Gamma^m D_m\hat{\lambda} - \frac{i}{2\cdot 5!}\bar{\hat{\lambda}}\Gamma^{mnpqr}F_{mnpqr}\hat{\lambda}\right). \tag{7.1}$$

We choose the following representation of the Γ-matrices

$$\begin{aligned} \Gamma^a &= \sigma^1\otimes I_4\otimes\gamma^a\,, \qquad \Gamma^\alpha = -\sigma^2\otimes\tau^\alpha\otimes I_4 \\ \{\Gamma_M,\Gamma_N\} &= 2\eta_{MN}\,, \qquad \{\gamma_a,\gamma_b\} = 2\eta_{ab}\,, \qquad \{\tau_\alpha,\tau_\beta\} = 2\delta_{\alpha\beta}\,. \end{aligned}$$

In this representation the matrix Γ_{11} is equal to

$$\Gamma_{11} = \Gamma^0\cdots\Gamma^9 = \begin{pmatrix} I_{16} & 0 \\ 0 & -I_{16}\end{pmatrix}. \tag{7.2}$$

The spinor is right handed, and the gravitino left-handed:

$$\hat{\lambda} = \frac{1}{2}(1-\Gamma_{11})\hat{\lambda} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \hat{\psi}_\mu = \frac{1}{2}(1+\Gamma_{11})\hat{\psi}_\mu = \begin{pmatrix} 0 \\ \psi_\mu \end{pmatrix}. \tag{7.3}$$

The action (7.1) becomes

$$S = \int d^{10}x\sqrt{-g}\,\bar{\lambda}\left(\gamma^a D_a + i\tau^\alpha D_\alpha + 1\right) l\,. \tag{7.4}$$

Expanding λ in spherical harmonics

$$\lambda = \sum_{k\geq 0}\left(l_k^+(x)\Xi_k^+(y) + l_k^-(x)\Xi_k^-(y)\right), \qquad \tau^\alpha D_\alpha\Xi_k^\pm = \mp i(k+\frac{5}{2})\Xi_k^\pm\,, \tag{7.5}$$

we obtain the five-dimensional action

$$S = \int d^5x\sqrt{-g_a}\sum_{k\geq 0}\left(\bar{\lambda}_k^+\left(\gamma^a D_a + k + \frac{7}{2}\right)l_k^+ + \bar{\lambda}_k^-\left(\gamma^a D_a - k - \frac{3}{2}\right)l_k^-\right). \tag{7.6}$$

There is no need to add any boundary term to this action, as the boundary conditions that we imposed in section 2 ensure that the classical action does not vanish on shell.

The ten-dimensional action for the gravitino is

$$S = \int d^{10}x\sqrt{-g}\left(\bar{\hat{\psi}}_m\Gamma^{mnp}D_n\hat{\psi}_p + \frac{i}{4\cdot 5!}\bar{\hat{\psi}}_m\Gamma^{mnp}\Gamma^{mnpqr}F_{mnpqr}\Gamma_n\hat{\psi}_p\right). \tag{7.7}$$

Rewriting this in terms of five-dimensional fields gives

$$\begin{aligned} S = \int d^{10}x\sqrt{-g}\Big(&\bar{\psi}_\mu\Big(\gamma^{\mu\nu\rho}D_\nu\psi_\rho - i\gamma^{\mu\nu}\tau^\alpha D_\nu\psi_\alpha + i\gamma^{\mu\nu}\tau^\alpha D_\alpha\psi_\nu + \gamma^\mu\tau^{\alpha\beta}D_\alpha\psi_\beta - \gamma^{\mu\nu}\psi_\nu\Big) + \\ &+\bar{\psi}_\alpha\Big(-i\tau^{\alpha\beta\gamma}D_\beta\psi_\gamma - i\gamma^{\mu\nu}\tau^\alpha D_\mu\psi_\nu + \gamma^\mu\tau^{\alpha\beta}D_\beta\psi_\mu - \gamma^\mu\tau^{\alpha\beta}D_\mu\psi_\beta + \tau^{\alpha\beta}\psi_\beta\Big)\Big). \end{aligned} \tag{7.8}$$

As in [13] we fix the local supersymmetries by transforming away all modes of $\tau \cdot \psi$ except the one proportional to the Killing spinor, on which the eigenvalue of $i\tau \cdot D_y$ is $5/2l$. We perform the decomposition

$$\psi_\mu = \varphi_\mu + \frac{D^T_\mu}{D \cdot D^T} D^T \cdot \psi + \frac{1}{d}\gamma_\mu \gamma \cdot \psi \,, \tag{7.9}$$

where $D^T_\mu = (\delta^\nu_\mu - \gamma_\mu \gamma^\nu/5)D_\nu$ is γ-transverse so that

$$\gamma \cdot \phi = D \cdot \phi = 0 \,. \tag{7.10}$$

If we put $\psi_1 = \sqrt{D \cdot D^T} D^T \cdot \psi$ and $\psi_2 = \gamma \cdot \psi$ then the change of variables $\psi_\mu \to (\phi_\mu, \psi_1, \psi_2)$ has a trivial jacobian. The expansion in spherical harmonics is given by

$$\begin{aligned}
\psi^T_\alpha &= \sum \psi^{I_T}(x)\Xi^{I_T}_{(\alpha)}(y) + \psi^{I_L}(x) D^T_\alpha \Xi(y) \\
\psi_\mu &= \sum \psi_\mu(x) \Xi^{I_L}(y) \\
\tau \cdot D \Xi^{I_T}_{(\alpha)} &= m^{I_T} \Xi^{I_T}_{(\alpha)} = \mp i \left(k + \frac{7}{2}\right) \Xi^{I_T}_{(\alpha)} \,, \qquad k \geq 1 \\
\tau \cdot D \Xi^{I_L} &= m^{I_L} \Xi^{I_L} = \mp i \left(k + \frac{5}{2}\right) \Xi^{I_L} \,, \qquad k \geq 1 \,.
\end{aligned} \tag{7.11}$$

The action for ϕ_μ decouples

$$S_\phi = \int d^{10}x \sqrt{-g}\, \bar{\phi}^\mu \left(\gamma \cdot D + \frac{(m^{I_L} - 1)}{l}\right) \phi_\mu \,, \tag{7.12}$$

but to remove the constraints (7.10) we must introduce Lagrange multiplier fields that are equivalent to introducing a pair of ghosts with masses $Ml = \sqrt{(m^{I_L} - 1)^2 + 4}$ [10]. Diagonalising the $(\psi_1, \psi_2, \psi^{I_T}, \psi^{I_L})$ system, we get two spinor fields that cancel the ghosts, and two spinor fields of mass $Ml = 3 + m^{I_L}$, $m^{I_T} - 1$, in agreement with the spectrum of [13]. Finally, in the case of the Killing spinor η, the shift

$$\psi_\mu \to \psi_\mu + \frac{5}{3} i \gamma_\mu \eta \tag{7.13}$$

gives the action for the massless gravitino and the mass $-11/2l$ for η.

8. Conclusions

We have shown that the AdS/CFT conjecture for IIB String theory/ $\mathcal{N}$=4 Super-Yang-Mills theory passes the stringent test of requiring that the Weyl anomalies of the two theories match at sub-leading order. This generalises the leading order test of Henningson and Skenderis but avoids perturbation theory in the metric by working with an exact solution to the Einstein equation in the bulk. At sub-leading order all the multiplets of IIB Supergravity contribute to the boundary theory Weyl anomaly an amount given by a universal formula that involves the four-dimensional heat-kernel for conformally covariant operators. The regularised sum of these contributions involves only those operators that appear in the boundary theory resulting in the matching of the anomalies. Our approach generalises to other AdS/CFT correspondences.

JHEP01(2004)013

References

[1] M. Henningson and K. Skenderis, *The holographic weyl anomaly*, *J. High Energy Phys.* **07** (1998) 023 [hep-th/9806087].

[2] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231.

[3] L. Bonora, P. Cotta-Ramusino and C. Reina, *Conformal anomaly and cohomology*, *Phys. Lett.* **B 126** (1983) 305.

[4] M.J. Duff, *Twenty years of the weyl anomaly*, *Class. and Quant. Grav.* **11** (1994) 1387 [hep-th/9308075] and refs. therein.

[5] P. Mansfield and D. Nolland, *Order $1/N^2$ test of the maldacena conjecture: cancellation of the one-loop weyl anomaly*, *Phys. Lett.* **B 495** (2000) 435 [hep-th/0005224].

[6] P. Mansfield and D. Nolland, *One-loop conformal anomalies from AdS/CFT in the schroedinger representation*, *J. High Energy Phys.* **07** (1999) 028 [hep-th/9906054].

[7] P. Mansfield and D. Nolland, *Test of gauged $N = 8$ SUGRA/$N = 1$ SYM duality at sub-leading order*, *Phys. Lett.* **B 515** (2001) 192 [hep-th/0105224].

[8] D. Nolland, *AdS/CFT boundary conditions, multi-trace perturbations and the c-theorem*, hep-th/0310169.

[9] D. Nolland, *Cancellation of divergences in $N = 4$ SYM/type-IIB supergravity correspondence*, *Phys. Lett.* **B 579** (2004) 235 [hep-th/0310201].

[10] D. Nolland, *Gravitinos in non-Ricci flat backgrounds*, *Phys. Lett.* **B 485** (2000) 308 [hep-th/0005267].

[11] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [hep-th/9905104].

[12] G.E. Arutyunov and S.A. Frolov, *Quadratic action for type-IIB supergravity on $AdS_5 \times S^5$*, *J. High Energy Phys.* **08** (1999) 024 [hep-th/9811106].

[13] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *The mass spectrum of chiral $N = 2$ $D = 10$ supergravity on S^5*, *Phys. Rev.* **D 32** (1985) 389.

[14] S.M. Christensen and M.J. Duff, *New gravitational index theorems and supertheorems*, *Nucl. Phys.* **B 154** (1979) 301.

[15] A.O. Barvinsky and G.A. Vilkovisky, *The generalized Schwinger-Dewitt technique in gauge theories and quantum gravity*, *Phys. Rept.* **119** (1985) 1.

[16] P. Mansfield, D. Nolland and T. Ueno, *Order $1/N^2$ test of the Maldacena conjecture, II. The full bulk one-loop contribution to the boundary weyl anomaly*, *Phys. Lett.* **B 565** (2003) 207 [hep-th/0208135].

[17] P. Mansfield, *Continuum strong coupling expansion of Yang-Mills theory: quark confinement and infrared slavery*, *Nucl. Phys.* **B 418** (1994) 113 [hep-th/9308116].

JHEP01(2004)013